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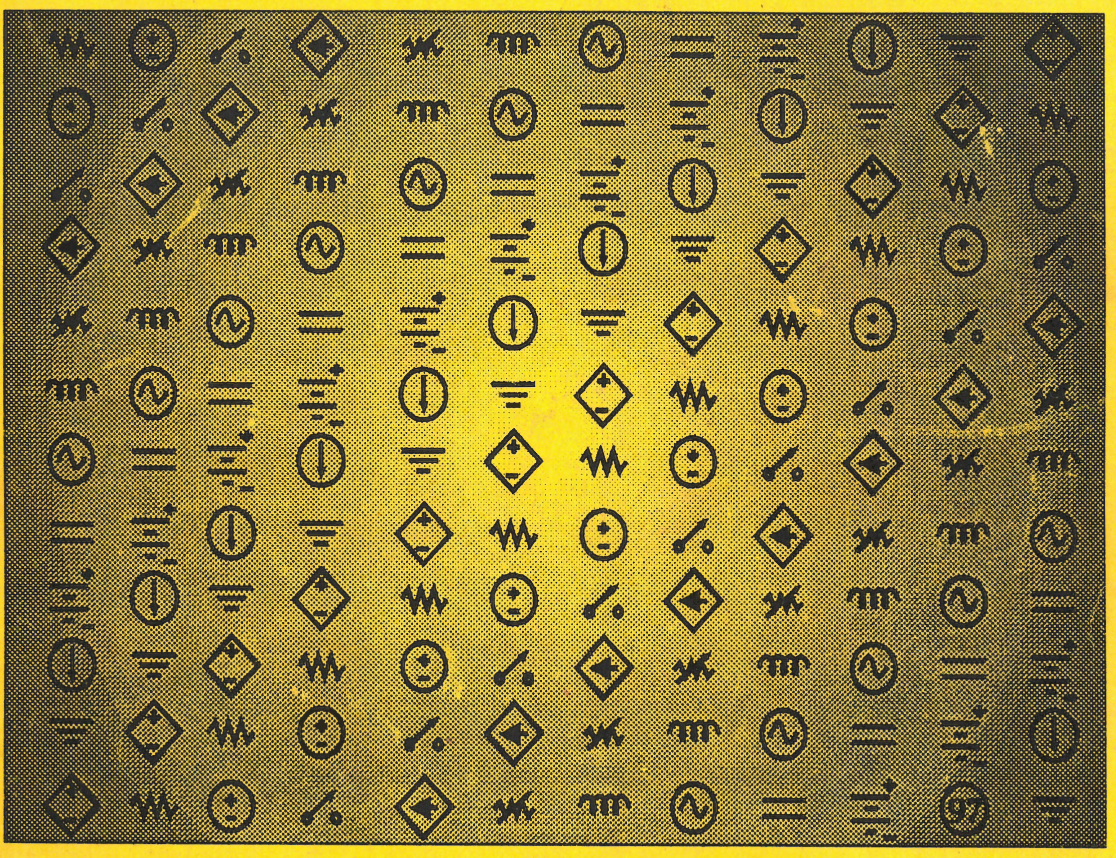
ELECTRICAL CIRCUIT ANALYSIS

A Student's Manual

G.M.Tattersfield

(1997 Edition)



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Mark Behr 4.107

All electrical components are made of smoke
When the smoke comes out, it stops working.

3

INTRODUCTION

Electrical Circuit Analysis is a part of the course given to Second Year Electrical and Electro-Mechanical Engineering Students at the University of Cape Town. This handbook, which is one of a series by the same author, has been expressly designed to accompany the Circuit Analysis module, which is entitled EEE221W Module A. The lecture series consists of 24 lectures (with two tests and a revision lecture), and the material for each of these lectures is presented here in detail. Examples of various circuit-analysis techniques have been left blank, ready to be filled in during lectures. Electrical Engineering subjects are frequently highly-detailed, and this module is no exception; so this book is written in the hope that students will not feel the need to scribble notes feverishly during lectures, but can instead be confident of finding most of the material conveniently laid out here.

Worksheets are given at the end of each chapter, and questions will be selected from each worksheet, to give a weekly exercise during the course. The tutorial questions are of a very similar standard to the selected worksheet questions, and the answers to these are supplied. It is envisaged that students will attempt all of the questions in each tutorial, confirming their techniques by comparison with the given answers, and will then present only the selected worksheet questions for assessment. The students receive rapid feedback, and in this way it is hoped to carry all of the students together through concepts of progressively-increasing difficulty. All of the questions are designed to make students think for themselves, and the emphasis in this module is for each student to gain in experience by solving a large number of problems.

The course aims to justify the study of calculus undergone by the students in the previous semester, and also to set the stage for the Engineering Mathematics (particularly the systematic solution of differential equations and the use of the Laplace transform) that they are due to encounter in the next semester. Recognition is given to the connectivity that exists between the various Engineering and Engineering Mathematics courses given at UCT, with a view to promoting a sense of direction and purpose in the students' approach to their 4-year Engineering degree course.

To the Student

This book provides you with most of the lecture material for the Circuit Analysis part of the First Semester module of your Second Year Electrical Engineering or Electro-Mechanical Engineering course (EEE221W or EEE225F for Science and ASPECT students). This part of the work in the first semester consists of 24 lectures, given concurrently with 24 lectures in Basic Electronics and 12 additional lectures, which are used as tutorial sessions, or else as extra time to cover certain areas that may require it. You will probably be in possession of a similar handbook, covering the Electronics side of the course. There are six practical sessions during the course, culminating in the Laboratory Test - you will receive details of these elsewhere.

The main aims in the circuit analysis course are to describe electrical circuits, to study some elementary aspects of their behaviour and to introduce some very important Electrical Engineering concepts. The treatment of electrical circuits is often fairly mathematical, and so it would be a good idea to make sure you are happy with the following topics before the course begins:

- Solution of simultaneous equations, including linear algebra methods
- Differentiation and integration of functions (first-year maths!)
- Complex numbers, up to Euler's Formula and de Moivre's Theorem

Make a point of attending all of the lectures: the course will move quite fast, and there is no real substitute for getting the material first hand. The notes in this manual are comprehensive, but they will not be enough without the lectures to give you a clear picture of the subject.

Your main study exercise is to complete the worksheets, of which there will be ten during the course. Each worksheet contains a good selection of questions covering the topics most recently lectured, and you are strongly recommended to attempt all of the tutorial questions, and then to give special attention to the worksheet questions, which are very exam-like!

It is advisable to discuss the questions in the tutorials among yourselves, but you must be *sure* you have independent and thorough understanding of each principle covered. This is to encourage each one of you to practise the skills of electrical circuit analysis, and to develop constructive collaboration.

Remember that the Lecturer and Tutors are available on a regular basis to assist you with any outstanding problems that you may have.

There is a two-hour examination at the end of the semester, on the work covered in all of the lectures and on all ten worksheets, and a specimen examination is included at the end of this book to show you the sort of questions that may be asked. There is no textbook to buy for this course, as the worksheets and tutorials in this student's manual will provide all the necessary practice. If you can do all the questions on the worksheets, understanding what you are doing, then you will have no trouble at all in the examination. If, however, you want to consult a textbook, a very popular one worldwide which is easily available here is:

Circuits, Devices and Systems by Ralph J. Smith, 4th Edition 1984, Published by John Wiley & Sons.

Welcome to the course! I hope you will enjoy this introduction to the theory upon which virtually all branches of Electrical Engineering are based.

G.M.T.

Contents

Chapter 1

Circuit Terminology

In Lecture C1 we cover:

- What is an electrical circuit?
- What does electrical circuit analysis mean?
- Analysis implies measurable quantities. What are they?
- What are the units for measurements in circuits?
- Basic conventions and definitions of circuit theory

1.1 The Building Blocks

Def 1: A (2-terminal) *element* is a purpose-built device with two ends that may be connected to other such devices. Examples of 2-terminal elements are resistors. capacitors, inductors, batteries and generators.

Def 2: An electric *circuit* or *network* is a collection of electrical elements interconnected in some specified way.

1.2 SI Units

To describe the role of an element in a circuit, we need measurements of quantities like *voltage* or *current.* These are measured in *SI Units:*

- Fundamental SI units are m, kg, s, C, K, cd

- Derived units include
  - the newton $N = kgm/s^2$ for force
  - the joule $J = Nm$ for work or energy
  - the watt $W = J/s$ for power

pico (p)

- Note the use of SI prefixes atto (a), femto (f), nano (n), micro (μ), milli (m), centi (c), deci (d), deca (D), hecto (h), kilo (k), mega (M), giga (G), tera (T), peta (P), eka (E) for convenience

1.3 Charge, Current and Voltage

A force exists which we ascribe to *charges.* Like charges repel and unlike charges attract. The *electron* carries a single (negative) charge.

Def 3: A *coulomb* has the charge of $6.24 * 10^{18}$ electrons. This is the SI unit of charge - a fundamental SI unit.

for positive charge

Def 4: *Electric current* is a movement of charge. $i(t) = dq/dt$. The SI unit of current is the *ampere.* Franklin guessed the direction wrong, so *conventional current* is opposite to the electron flow.

The charge going onto an element is $q_T = q(t) - q(t_0) = \int_{t_0}^{t} idt$. Currents as a function of time may be DC, AC, exponential, sawtooth etc.

In a piece of metal, charges move randomly, unless some work is done by an *electromotive force (EMF)* to produce coherent movement or current.

Def 5: The work in joules done in moving each coulomb of charge across the element is called the *voltage across the element.* $1Volt = 1J/C$ is the SI unit.

1.4 Energy and Power

Polarity markings on elements make the direction of voltage rise clear, and show whether energy is supplied *to* or *by* an element in a circuit. Great care with signs is needed. Double subscript notation may help.

Energy $w = vq$, so the energy to move a small charge is $\Delta w = v\Delta q$. If this occurs in time Δt, then $\Delta w/\Delta t = v\Delta q/\Delta t$, and in the limit we obtain

$dw/dt = vdq/dt = vi$. Power is the rate of transfer of energy, so $p = vi$. Note the consistency of the units.

If v and i are time-varying, p is called the *instantaneous power*. The total power absorbed or delivered from time t_0 to t is $w(t) - w(t_0) = \int_{t_0}^{t} vidt$. More generally, $w(t) = \int_{-\infty}^{t} vidt$.

1.5 Active and Passive Elements

Def 6: A *passive* element (e.g. a resistor, capacitor, inductor) always absorbs energy from the rest of the circuit. An *active* element (e.g. a generator, transistor, battery, etc.) may deliver energy to the rest of the circuit. Note, however, that active sources may absorb energy (e.g. a charging battery).

An *ideal independent voltage source* always has a specified voltage between its terminals. Note the different symbols if the specified voltage is time-varying or constant. We sometimes use *dependent* voltage sources, for which another symbol exists.

An *ideal independent current source* always has a specified current flowing through it.

Note that *ideal* sources are mathematical devices and that they imply infinite power. Nevertheless, they are valuable approximations to *real* sources.

1.6 Electrical Circuit Analysis

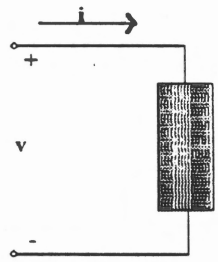
What is *circuit analysis*? Given a circuit and its *input*, circuit analysis aims to find the circuit's *output*. This may amount to finding only one voltage or current somewhere in the circuit, or it may mean *solving* for many such quantities in the circuit.

Contrast later courses in which you are given the input and the output, and asked to find the circuit. This is known as *synthesis* or *design*.

TUTORIAL 1

1.1
(a) How many electrons are represented by a charge of 0.48077pC?
(about 3 000 000)
(b) The total charge entering the terminal of an element is given by $q(t) = 2t^3 - 4t$ mC. Find the current i at $t = 0$ and at $t = 2$ s.
(-4mA, 20mA)
(c) The current entering a terminal is given by $i = 1 + \pi \sin(2\pi t)$ A. Find the total charge entering the terminal between $t = 0$ and $t = 1.5$ s.
(2.5 C)

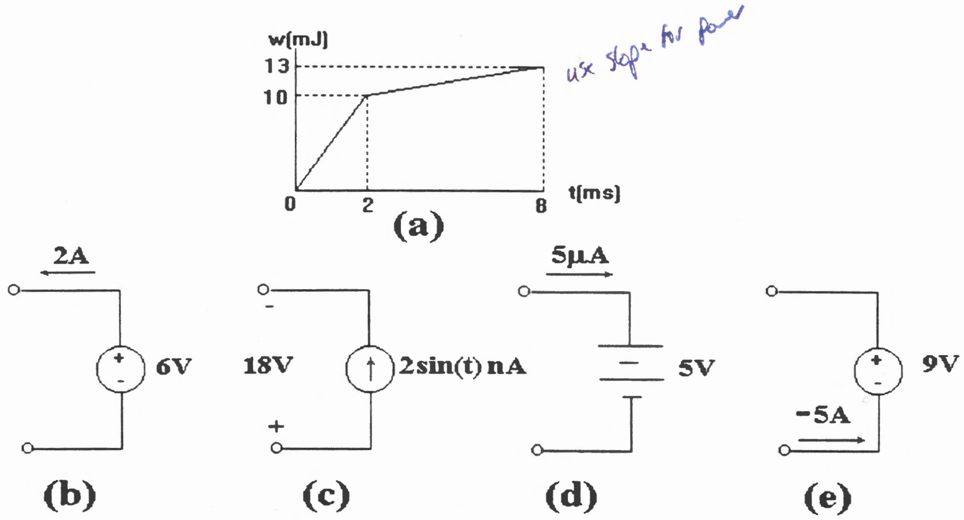
1.2
With reference to the element shown:



(a) Find v if $i = 8$ mA and the element is
(i) absorbing power at $p = 40$ mW
(ii) delivering power at $p = 16$ mW
(5V, -2V)
(b) If $i = 5$ A and $v = 12$ V, find
(i) the power absorbed by the element
(ii) the energy delivered to the element between 2 and 4 s.
(60W, 120J)

1.3
With reference to Figure 1.1 (a) to (e):
(a) A two-terminal element absorbs energy w as shown in fig (a). If the voltage across the element is $v = 30\cos(1000\pi t)$ V, find the current entering the positive terminal of the element at $t = 1$ ms and at $t = 4$ ms.
(-167mA. 16.7mA)
(b) Find the power being supplied by each of the sources in figs (b) to (e).
(12W, -36sin(t)nW, -25μW, -45W)

1.4
The terminal voltage of a voltage source is $v = 6\sin(2t)$ V. If the charge leaving the positive terminal is $q = -2\cos(2t)$ mC, find
(a) the power supplied by the source at any time
(b) the energy supplied by the source between 0 and t s.
($24\sin^2(2t)$ mW, $12t - 3\sin(4t)$ mJ)



use slope for power

Figure 1.1: Figures (a) to (e) for Question 1.3

1.5
The power delivered to an element is $p = 24e^{-8t}$ mW and the charge entering the positive terminal is $q = 2 - 2e^{-4t}$ mC. Find:
(a) the voltage across the element
(b) the energy delivered to the element between $t = 0$ and $t = 0.25$ s.
($3e^{-4t}$ V, 2.594 mJ)

1.6
With reference to Figure 1.2:
(a) If f(t) is the current in amperes entering the positive terminal (+) of an element, find:
    (i) the total charge that enters the element from $t = 4$ to $t = 12$ s
    (ii) the current at $t = 5$s, at $t = 8$s and at $t = 11$s.
(b) If f(t) is the charge that has entered the element in coulombs, find:
    (i) the total charge entering the element between $t = 2$s and $t = 10$s
    (ii) the current *leaving* the positive terminal at $t = 6.5$s and at $t = 11$s.
(c) If the voltage across the element is +12V, find at $t = 1, 5, 8$ *and* 10 s:
    (i) the power delivered to the element if f(t) is current entering +
    (ii) the power delivered to the element if f(t) is charge entering +
(32C. 6A. 5A. 667mA; -1.67C, -2A, 667mA;
18W. 72W. 60W. 16W; 18W, 0, -36W, -8W)

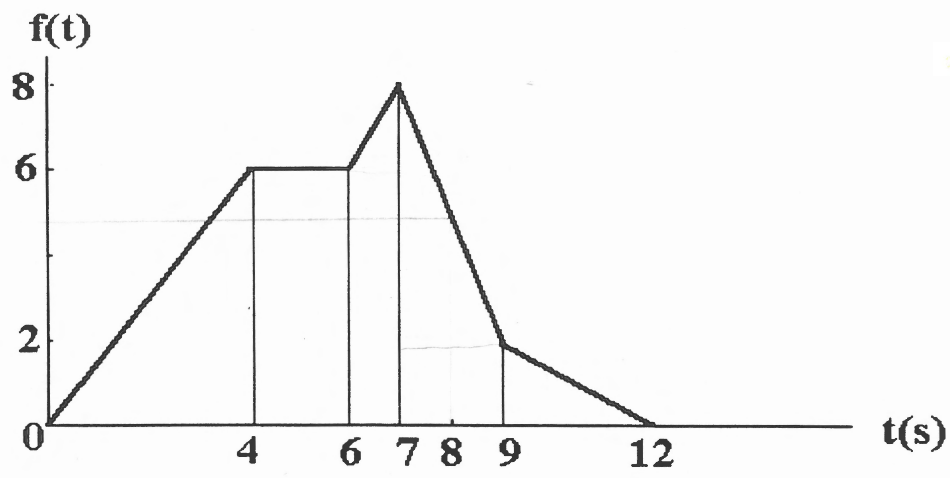


Figure 1.2: Graph referred to in Question 1.6

1.7
(a) Find the instantaneous power delivered to an element at $t = 4$ ms if the charge entering the positive terminal is $q = -12\cos(125\pi t)$ mC, and the voltage is $v = 8\sin(125\pi t)$ V.
(b) Find the energy delivered to the element between $t = 4$ms and $t = 8$ms.
(37.7W, 75.4mJ)

1.8
If the function graphed in Figure 1.3 is the voltage across an element versus the time in milliseconds, and the current entering the positive terminal is $i = 3\frac{dv}{dt}\mu$A, find the instantaneous power delivered to the element both at $t = 1$ ms and at $t = 7$ ms.
(18.8mW, 24mW)

1.9
The power delivered to an element is $p = 12\sin(4t)$ W and the voltage is $v = 4\sin(2t)$ V. Find the current entering the positive terminal and the charge delivered to the element between 0 and $\pi/4$ s.
($6\cos(2t)$ A, 3C)

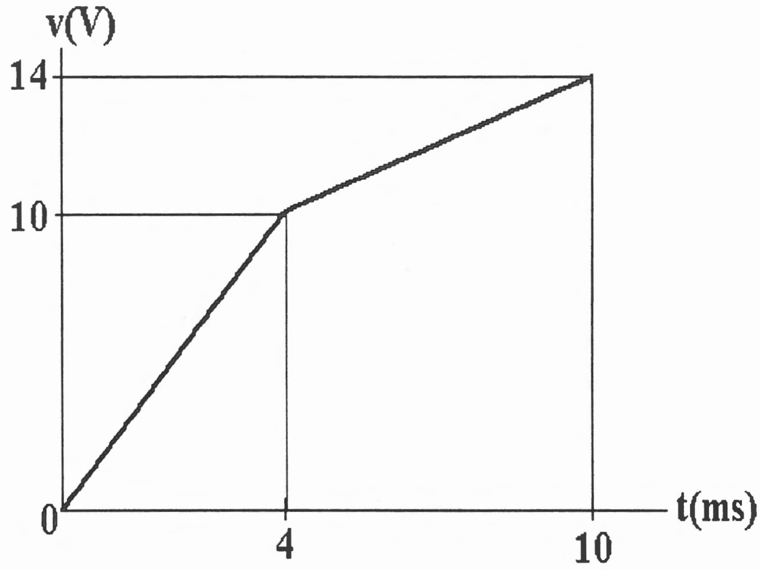


Figure 1.3: Graph referred to in Question 1.8

1.10

If the voltage across an element in a circuit is $v = 6e^{-3t}$ V and the current is $i = 2\frac{dv}{dt}$ A out of the positive terminal,

(a) find the power delivered by the element to the circuit at time t

(b) find the charge delivered to the circuit between $t = 0$ and $t = 4$ s.

(c) Given that the element is a battery, comment on your answers.

($-216e^{-6t}$ W, -12C, power and charge are being delivered *to* the battery)

group letter (E)

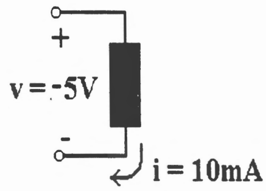
WORKSHEET 1

1.A

(a) Given that a flow of 6.24×10^{18} electrons per second represents a current of 1A, what charge is represented by 10 million electrons?

(b) The current entering a terminal is given by $i = 2 - 3\pi\cos(3\pi t)$ A. Find the total charge entering the terminal from $t = 0$ to $t = 4s$.

(c) With reference to the element shown:

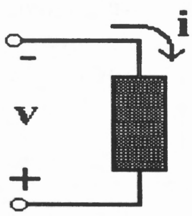


(i) Find the power *absorbed* by the element.
(ii) Find the energy *delivered* by the element between $t = 1s$ and $t = 3s$.

(d) The terminal voltage of a voltage source is $v(t) = 6\sin(2t)$ V. If the charge leaving the positive terminal is $q(t) = -2\cos(2t)$ mC, find:

(i) the power supplied by the source at any time t.
(ii) the power supplied by the source at $t = \frac{1}{2}$ s.

1.B



(a) With reference to the element shown, find the power absorbed by the element, if $i = 5$A dc and $v = 24$V dc.

(b) An element receives into its positive terminal a time-varying charge $q(t) = 3 - 3e^{-4t}$mC. In doing so, it absorbs power $p(t) = 24e^{-8t}$mW. Find:

(i) the voltage across the element as a function of time,

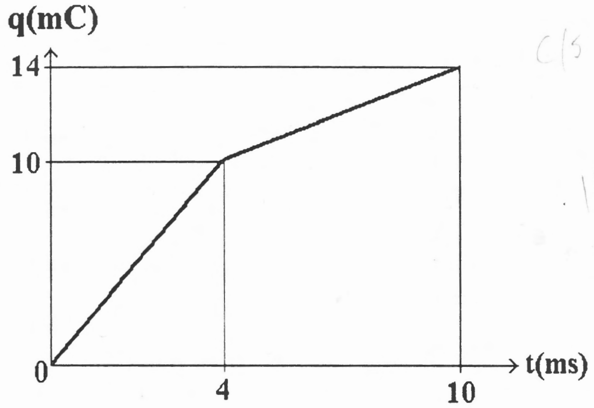
(ii) the energy delivered to the element between $t = 0$ and $t = 0.1$s.

1.C

The graph below shows the charge in millicoulombs that has entered the positive terminal of an element, as a function of the time in milliseconds from $t = 0$. The voltage across the element is $v(t) = 6t$ V. Find:

(i) the total charge delivered to the element over the first 10 ms;

(ii) the total energy delivered to the element over the first 10 ms.



1.D*

(a) Given that 100 million electrons represents a charge of 16.026 pC, how many electrons per second pass a point in a circuit if a current of 3A flows there?

(b) The current entering an element's positive terminal is $i = -4e^{-2t}$ A, and the voltage across the element is $v = 4i$ V. Find the energy delivered to the element between $t = 0$ and $t = 1$ s, answering in joules to the nearest millijoule.

(c) The total positive charge that has entered the positive terminal of an element is given as a function of time by $q(t) = 5\sin(7000t)\,\mu$C. Find the conventional current *leaving* the positive terminal at $t = 4$ ms.

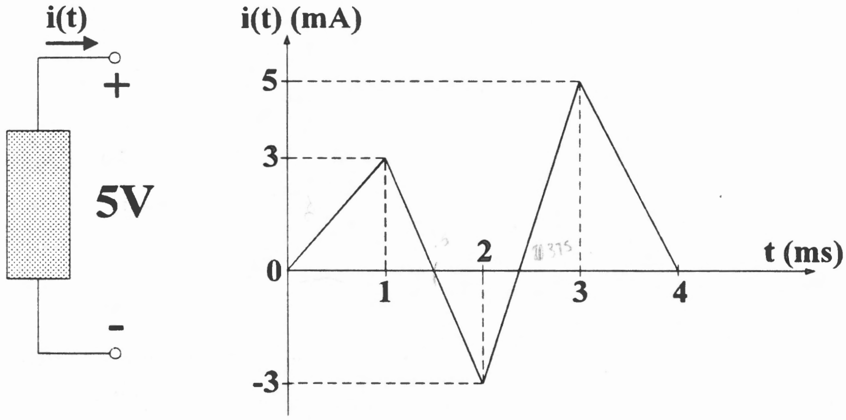
(d) Find the power absorbed by an element whose positive terminal is at a voltage of 8V with respect to the negative terminal, and from whose negative terminal a current of -3A flows.

1.E*

The graph below gives the current in milliamperes that flows out of the positive terminal of the element shown, as a function of time in milliseconds.

(a) Find the net charge that leaves the positive terminal of the element between $t = 0$ and $t = 4\,\text{ms}$

(b) At t= 2 ms, what is the value of the instantaneous power being delivered by the element to any circuitry attached to its terminals?



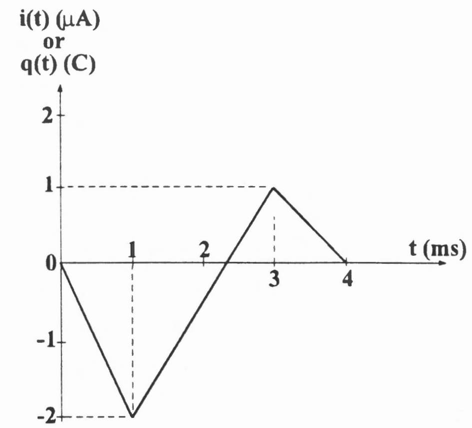
1.F

(a) Draw a neat circuit diagram for the circuit described below, including all necessary symbols and labels:

"A 1A independent current source, in parallel with a dependent voltage source of value $4v_x$, drives a 2Ω resistor which is connected in series with a parallel combination of a 3Ω resistor and a 5Ω resistor down to ground. The voltage drop across the series resistor is v_x."

(b) (i) If the graph on the next page shows the current in μA leaving the positive terminal of an element as a function of time, find the charge on the positive terminal after 2ms, given that the positive terminal carried an initial charge of 750pC at time $t = 0$.

(ii) If the graph on the next page represents instead the charge in coulombs on the positive terminal of an element, find the power being delivered to the element at $t = 3.5$ms, given that the voltage drop across the element is 5kV.



(c) (i) A manufacturer opens a consignment of resistors to find that each resistor is marked:

RED PURPLE YELLOW SILVER

What is the lowest value of resistor that the manufacturer could expect to find in the consignment?

(ii) Explain why the manufacturer might prefer to use some other resistors that he has in stock, which are marked:

RED PURPLE BLACK ORANGE BROWN

Chapter 2

Basic Laws for Resistive Networks

In Lectures C2 and C3 we cover:

- The simplest and commonest circuit element, the resistor
- Ohm's Law
- The Current and Voltage Laws of Gustav Kirchhoff
- Series resistance and voltage dividers
- Parallel resistance and current dividers
- A brief look at ammeters, voltmeters and ohmmeters

2.1 Resistors and Ohm's Law

Metals and other conducting materials are made up of a *lattice* of atoms, which impedes or *resists* the motion of electrons through it. The more it does so, the higher the *resistance* of the material.

You can buy resistors of accurately-measured resistance. They are colour-coded, and you should know the code. Usually there are four coloured lines a,b,c,d and the resistance is "$(10a + b) * 10^c \pm d\%$", where a,b,c are found

from:
black 0, brown 1, red 2, orange 3, yellow 4, green 5, blue 6, purple 7, grey 8, white 9. Additionally c may be silver -2, or gold -1. Finally d may be silver ($\pm 10\%$), or gold ($\pm 5\%$), and is called the *tolerance.* 1% Brown 2% red.

The more current flows through a resistive element, the higher the voltage across its terminals. *Ohms Law* ($v = iR$) states that the two are in fact proportional, and the constant of proportionality is the *resistance*, R. The SI unit of resistance is the *ohm*, and $1\Omega = 1V/A$.

Note that Ohm's Law holds for a resistor only over some range of currents and voltages and eventually breaks down. So long as it holds, we call the resistor a *linear resistor.* Real resistors (e.g. a light bulb filament) are non-linear, The resistors we work with in our theory are assumed to be *ideal*, however.

An ideal zero resistance is called a *short circuit*, and an ideal infinite resistance is called an *open circuit.*

2.2 Power Dissipation in Resistors

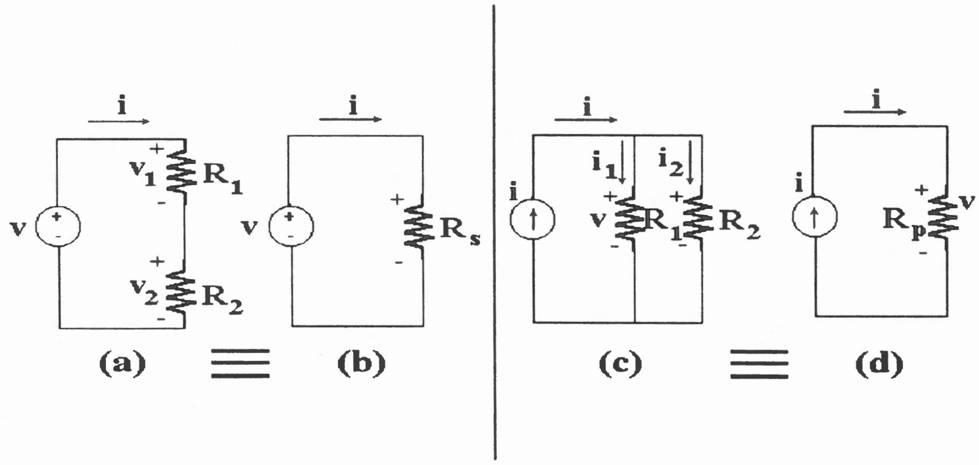
When a charge (q) passes through a resistor, and falls through a voltage (v), it expends energy (qv). The resistor absorbs the energy and heats up. The rate of this energy dissipation is the instantaneous power absorbed, which we know is $p(t) = v(t)i(t)$; so using Ohm's Law, *in the case of a resistor*, $p(t) = i^2(t)R = v^2(t)/R$.

The *power rating* of a resistor is the maximum power it can absorb without damage - usually about 1/4 W or 1/2 W for the ones in the lab.

Some people like to talk about the *conductance* G instead of resistance R. $G = 1/R$, and has SI units of *siemens* $S = 1/\Omega$ or *mho* $\mho = 1/\Omega$.

2.3 Kirchhoff's Laws

Two or more circuit elements meet at a *node.* Note: the perfect conductors that are drawn to join elements are all part of the node. Since current can't just be created from nothing, *the algebraic sum of the currents entering any node (or leaving any node) is zero.* This is KCL: $\sum_{n=1}^{N} i_n = 0$



Likewise, since energy can't be created, *the algebraic sum of the voltages around any closed loop is zero.* This is KVL: $\sum_{n=1}^{N} v_n = 0$. The formal proof of Kirchhoff's Laws is not given in this course.

2.4 Series Connection and Voltage Division

Elements carrying exactly the same current in a circuit are said to be in *series* (e.g. R_1 and R_2 in Figure (a)). By KVL, $v - v_1 - v_2 = 0$, so $v = v_1 + v_2$. By Ohm's Law, $v_1 = R_1 i$ and $v_2 = R_2 i$, and so $v = R_1 i + R_2 i$. From this, $i = v/(R_1 + R_2) = v/R_s$, where R_s is the *equivalent series resistance* shown in Figure (b). Thus $R_s = R_1 + R_2$.

The algebra above immediately gives $v_1 = \frac{R_1}{R_1+R_2} v$ and $v_2 = \frac{R_2}{R_1+R_2} v$, which are the very important rules for *voltage division.* To generalise, given N resistors in series, $R_s = R_1 + R_2 + \ldots + R_k + \ldots + R_N = \sum_{n=1}^{N} R_n$, and voltage division applies with $v_k = \frac{R_k}{R_s} v$.

NB

Furthermore, the power delivered to the resistors is
$p = \frac{v_1^2}{R_1} + \frac{v_2^2}{R_2} + \ldots + \frac{v_k^2}{R_k} + \ldots + \frac{v_N^2}{R_N} = \frac{R_1 v^2}{R_s^2} + \frac{R_2 v^2}{R_s^2} + \ldots + \frac{R_k v^2}{R_s^2} + \ldots + \frac{R_N v^2}{R_s^2}$
$= v^2/R_s = vi$, which is the power delivered by the source. This important result is the *conservation of power.*

$$p = \left(\frac{R_1}{R_s} v\right)^2 \frac{1}{R_1} + \ldots +$$
$$= \frac{R_1 V^2}{R_s^2} + \frac{R_2 v^2}{R_s^2} + \frac{R_3 v^2}{R_s^2} + \ldots + \frac{R_n v^2}{R_s^2} = \frac{R_s v^2}{R_s^2} = \frac{V^2}{R_s}$$
$$= \frac{v^2}{R_s}$$

2.5 Parallel Connection and Current Division

Elements which have a common voltage across them (i.e. whose terminals meet at two corresponding nodes) are said to be in *parallel* (e.g. R_1 and R_2 in Figure (c)). By KCL, $i = i_1 + i_2$, and by Ohm's Law, $i_1 = v/R_1$ and $i_2 = v/R_2$. Hence $i = v/R_1 + v/R_2 = \frac{R_1+R_2}{R_1R_2}v$. The *equivalent parallel resistance* shown in Figure (d) is therefore $R_p = \frac{R_1R_2}{R_1+R_2}$, which may be remembered as "product over sum" in the 2-resistor case.

For n resistors in parallel

$i_k = \frac{R_P}{R_k} i$

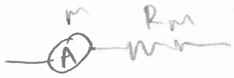
prove this

Since $v = \frac{R_1R_2}{R_1+R_2}i$, the above algebra gives $i_1 = \frac{R_2}{R_1+R_2}i$ and $i_2 = \frac{R_1}{R_1+R_2}i$. This is *current division*: note how it differs from voltage division. The algebra also yields $1/R_p = 1/R_1 + 1/R_2$ or $G_p = G_1 + G_2$ (conductance). For N resistors in parallel this generalises to $1/R_p = 1/R_1 + 1/R_2 + \ldots + 1/R_k + \ldots + 1/R_N = \sum_{n=1}^{N} 1/R_n$, with any particular branch current i_k being found as $i_k = R_p i/R_k$.

You can prove conservation of power for N resistors in parallel: the total power dissipated in the resistors is $p_1 + p_2 + \ldots + p_k + \ldots + p_N = R_1 i_1^2 + R_2 i_2^2 + \ldots + R_k i_k^2 + \ldots + R_N i_N^2 = i^2 R_p^2 (1/R_1 + 1/R_2 + \ldots + 1/R_k + \ldots + 1/R_N) = i^2 R_p = iv$, which is the power supplied.

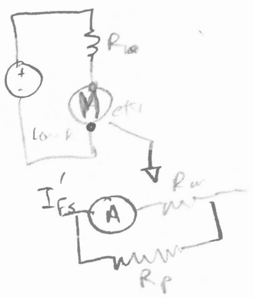
2.6 Ammeters, Voltmeters and Ohmmeters

In practice, we use meters to investigate the properties of a circuit. A very simple meter is the *d'Arsonval meter*, which consists of a pointer attached to a coil, through which a direct current passes, causing the pointer to move in proportion to the size of the current. We model the d'Arsonval meter as an ideal ammeter M, with a small associated series resistance R_M. Full-scale deflection of the pointer occurs if current I_{FS} flows in the meter.

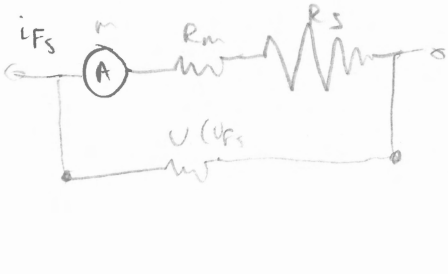


An *ideal ammeter* measures the current flowing through its terminals but has zero voltage across them - it has zero resistance. In practice, to measure currents above I_{FS}, a parallel resistance R_P is added to the d'Arsonval meter, which carries away current in excess of I_{FS}. The new full-scale current is I'_{FS}, and, by current division, $I_{FS} = \frac{R_P}{R_P+R_M}I'_{FS}$. From this, $R_P = \frac{R_M I_{FS}}{I'_{FS} - I_{FS}}$.

with shunt resistor, have to recalibrate scale



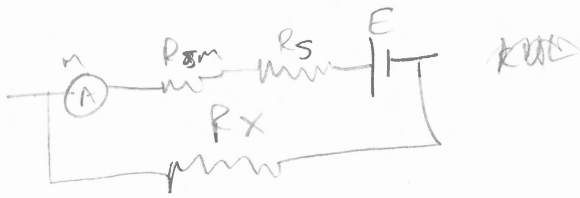
An *ideal voltmeter* measures the voltage across its terminals but has no terminal current - it has infinite resistance. A real voltmeter can be made from a d'Arsonval meter with extra series resistance R_S. If V_{FS} is the reading when I_{FS} flows in the meter, KVL gives $V_{FS} - R_S I_{FS} - R_M I_{FS} = 0$, and

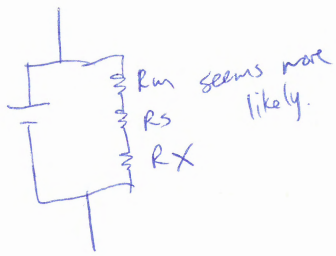


so the necessary resistor to add is $R_S = \frac{V_{FS}}{I_{FS}} - R_M$.

An *ideal ohmmeter* measures the resistance between its terminals but delivers zero power to the resistor being measured. A simple practical ohmmeter is a d'Arsonval meter with a resistor R_S and a battery E in series. If this circuit causes current i to flow in the unknown resistor R_X, then KVL gives $E - i(R_S + R_M + R_X) = 0$, so $R_X = \frac{E}{i} - (R_S + R_M)$.

Notice the pattern of our circuit analysis so far: find an equation and then solve for the unknown. If we have 2 (or more) unknowns, then we will need to have 2 (or more) simultaneous equations to find the unknown quantities. We now look at some convenient ways of deriving the simultaneous equations necessary for solving several unknowns in a single circuit.





TUTORIAL 2

2.1

(a) The terminal voltage of a 20kΩ resistor is 100V. Find:

(i) the conductance

(ii) the terminal current

(iii) the minimum wattage of the resistor

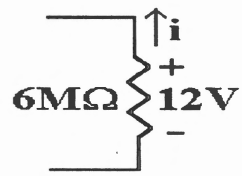
(50$\mu\mho$, 5mA, 500mW)

(b) The instantaneous power absorbed by a resistor is $4\sin^2 377t$W. If the current is $40\sin 377t$mA, find v and R.

($100\sin 377t$V, 2.5kΩ)

(c) Find i and the power delivered to the resistor.

(-2μA, 24μW)



2.2

(a) Find i and v_{ab} in Figure 2.1(a).

(-1A, 62V)

(b) Find v and i in Figure 2.1(b).

(17V, 3A)

2.3

(a) In Figure 2.2(a), find (i) the equivalent resistance seen by the source, (ii) the current i, (iii) the power delivered by the source, (iv) v, (v) v_2, and (vi) the minimum wattage required for the 6Ω resistor.

(b) In Figure 2.2(b), find the equivalent resistance seen by the source, and use the result to find i, i_1 and v.

3 15

(24Ω, 500mA, 6W, 5V, ~~-4V~~, 1.5W; 8Ω, ~~6~~A, 2.5A, ~~30~~V)

4V -5V

2.4

(a) A load requires 3A and absorbs 48W. If only a 5A current source is available, find the required resistance to place in parallel with the load.

(8Ω)

(b) Find the equivalent resistance seen by the source, and find the current i in Figure 2.3.

(10Ω, 2A)

2.5

(a) In Figure 2.4(a), find v_{ab} and the power delivered by the 5V source.

(5V, 500mW)

(b) In Figure 2.4(b), find R and construct an equivalent circuit which has one current source and a single resistor.

(R = 20Ω, $i = 3\sin tA$, 10Ω resistor)

KVL: $V_{ab} - 8 - 4 - 56 + 6 = 0$

$\therefore V_{ab} = 62V$

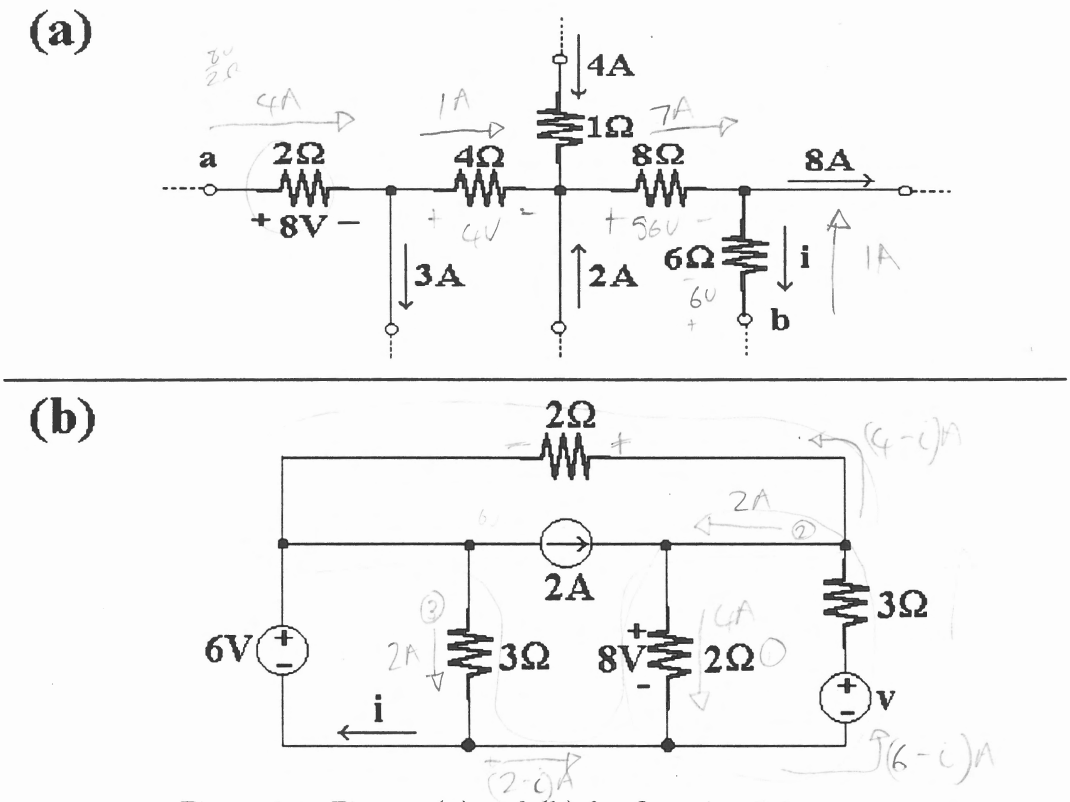


Figure 2.1: Figures (a) and (b) for Question 2.2

KVL $8V - 2(4 - i) - 6 = 0$ $i = 3A$

$V - 9V - 8 = 0$ $V = 17V$

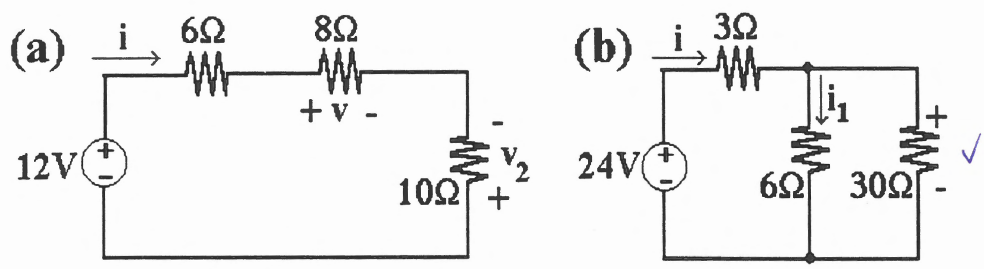


Figure 2.2: Figures (a) and (b) for Question 2.3

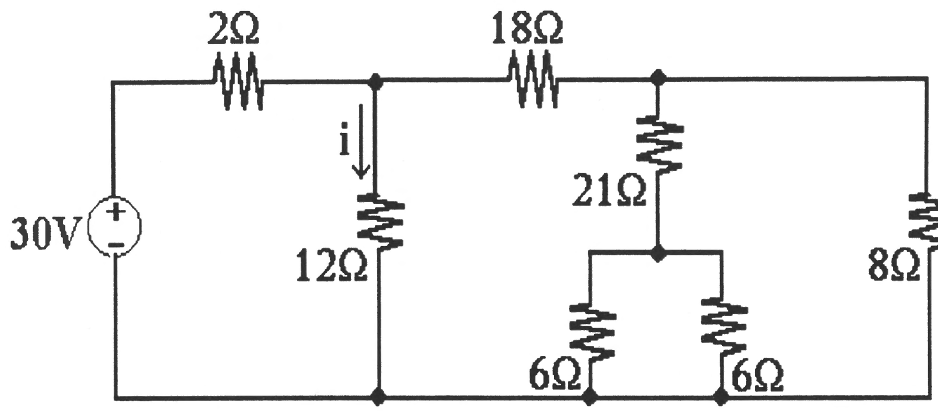


Figure 2.3: Figure for Question 2.4

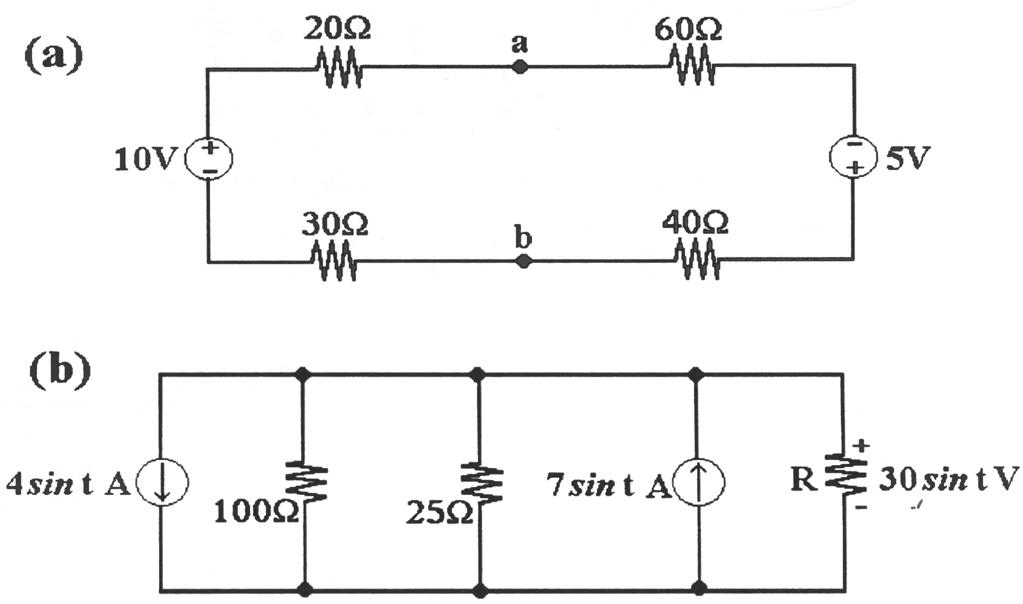


Figure 2.4: Figures (a) and (b) for Question 2.5

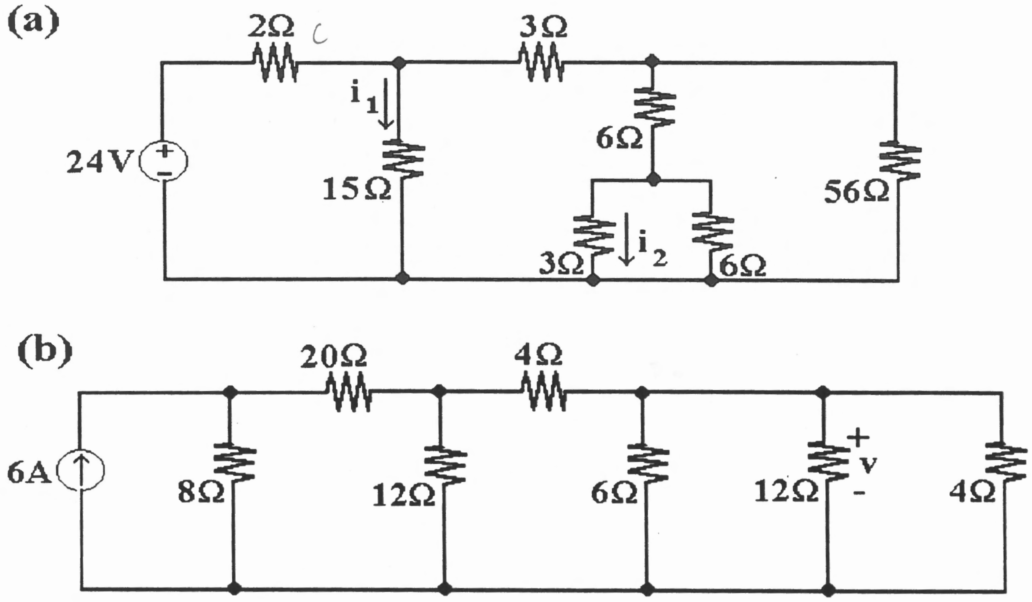


Figure 2.5: Figures (a) and (b) for Question 2.6

2.6

(a) Find i_1 and i_2 in the circuit of Figure 2.5(a).
(b) Find v and the power delivered by the source in the circuit of Figure 2.5(b).
1.2A 1.05 2 2/6
($R_{eq} = 8\Omega$, ~~800mA~~, ~~700mA~~; $R_{eq} = 6\Omega$, ~~4~~V, ~~864~~W)

2.7

(a) A d'Arsonval meter has I_{FS} = 1mA and R_M = 50Ω. Determine the parallel resistance R_P that should be connected to the meter, so that I'_{FS} is (i) 1.0 mA, (ii) 10 mA, (iii) 100 mA.
(∞, 5.556Ω, 0.505Ω) 0.505 Ω
(b) Determine the necessary series resistance R_S and the Ω/V rating for a voltmeter to have a full-scale voltage of 100V, if the voltmeter is constructed from a d'Arsonval meter with (i) R_M = 100Ω and I_{FS} = 50μA, and (ii) R_M = 50Ω and I_{FS} = 1mA.
(~~2M~~Ω, 20kΩ/V; 100kΩ, 1kΩ/V)
~~∞ 5.56 Ω 505mΩ~~

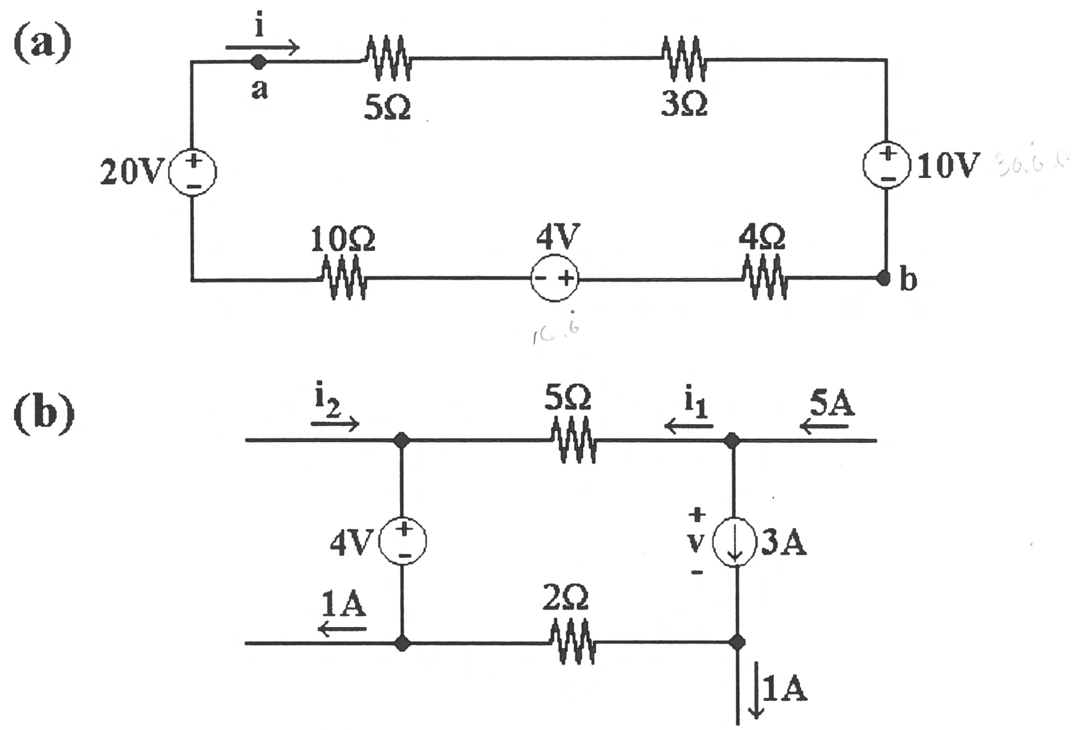


Figure 2.6: Figures (a) and (b) for Question 2.8

ask about this one

2.8
(a) In the circuit of Figure 2.6(a), find i, v_{ab}, and an equivalent circuit containing a single source and only one resistor.
(b) Find i_1, i_2 and v in the circuit of Figure 2.6(b).
(273mA, 13.3V; 2A, -3A, 10V)
? 12.182

2.9
(a) A d'Arsonval meter has a full-scale current of 1 mA and a resistance of 4.8Ω. If a 0.2Ω parallel resistor is connected across the meter, what is I'_{FS}, and what voltage occurs across the meter?
(b) A 40 000 Ω/V voltmeter has a full-scale voltage of 120V. What current flows in the meter when measuring 80V?
(~~30mA~~, 2.9mV; $R_s = \cancel{3.6}$MΩ, ~~25~~μA)
2.5mA 4.8mV 4.8 16.7

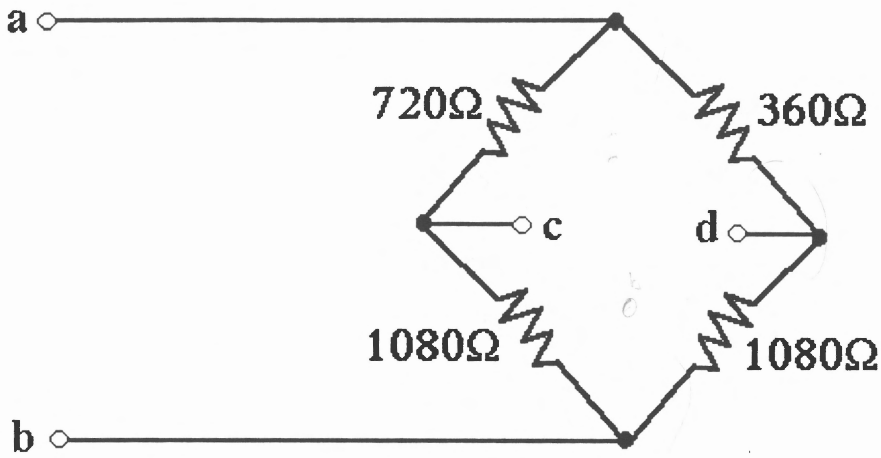


Figure 2.7: Figure for Question 2.10

2.10 Tafts most favourite question in whole course

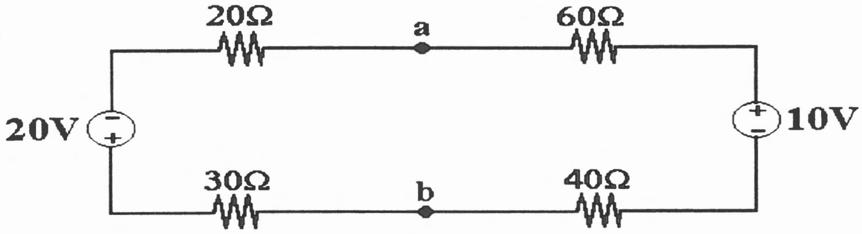
With reference to figure 2.7:
(a) Find the equivalent resistance looking into terminals a-b if terminals c-d are open, and if terminals c-d are shorted together.
(b) Find the equivalent resistance looking into terminals c-d if terminals a-b are open, and if terminals a-b are shorted together.
(800Ω, 780Ω; 720Ω, 702Ω)

WORKSHEET 2

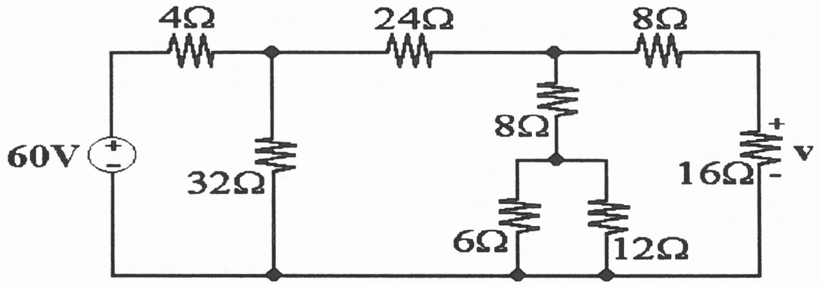
2.A

(a) Find v_{ab} in circuit (a) below.

(a)



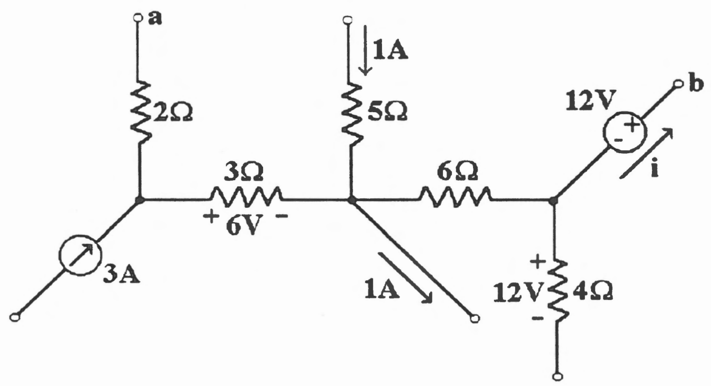
(b)



(b) Find the equivalent resistance seen by the 60V source in circuit (b) above, and *hence* calculate the voltage v across the 16Ω resistor.

2.B

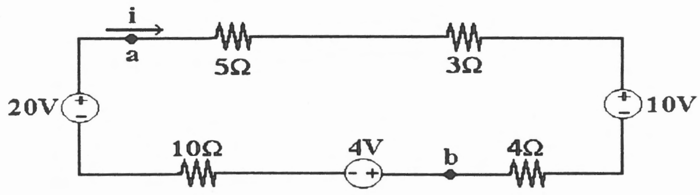
Find i and v_{ab} in the circuit shown below.



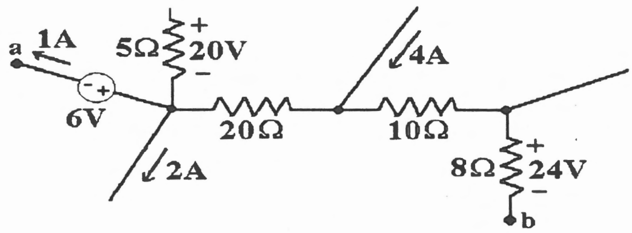
2.C*

(a) Find v_{ab} in each of the two circuits given below

(i)



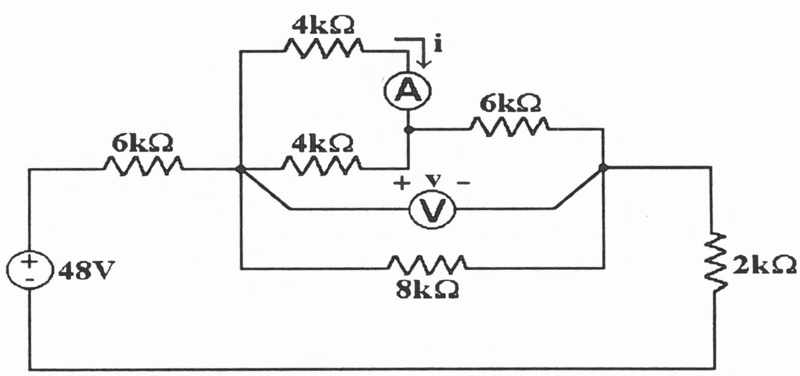
(ii)



(b) What values of voltage v and current i will be registered on the voltmeter and the ammeter shown in the circuit below?

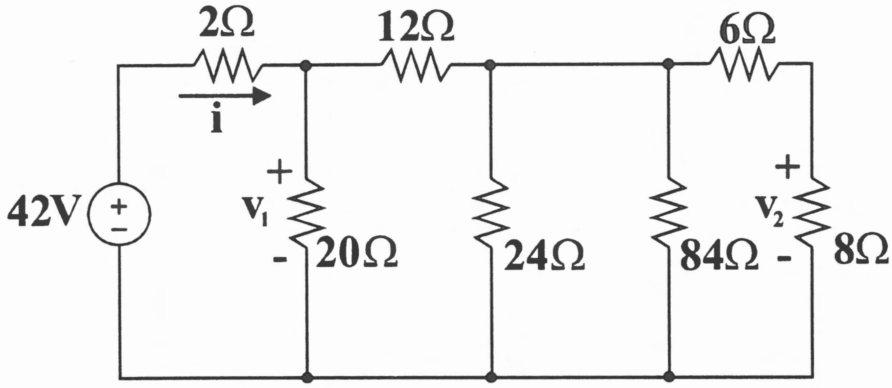
(c) It is proposed to use D'Arsonval meters to make the measurements of voltage and current. Each meter has an internal resistance of 8Ω and full-scale current of 1mA. The voltmeter should read up to 40V, and the ammeter should read up to 5mA.

What additional resistances are needed in each meter, and how should they be connected?



2.D

(a) Find the equivalent resistance seen by the 42V source in the circuit



shown here, and *hence* calculate the values of i, v_1 and v_2.

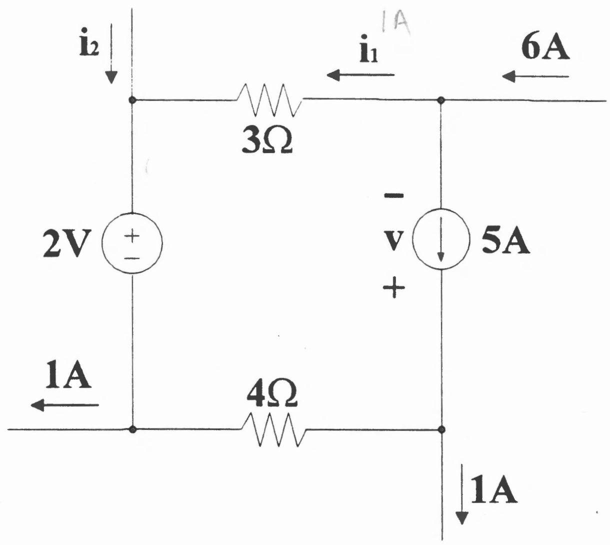
(b) Show by means of a sketch how a d'Arsonval meter may be used as an ammeter. If a d'Arsonval meter has a full-scale current of 1mA and an internal resistance of 2.7Ω, what value of resistance should be added, and how should it be connected, to create an ammeter with a full-scale deflection of 10mA?

2.E*

(a) Find i_1, i_2 and v in the circuit on the top of the next page.

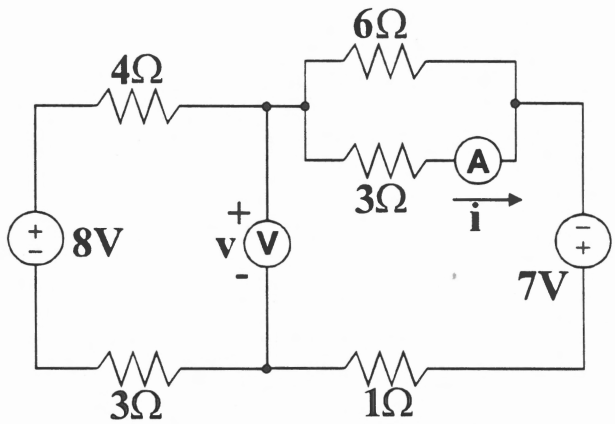
(b) A current divider consists of the parallel connection of four resistors of values 20 kΩ, 40 kΩ, 60 kΩ and 120 kΩ.

Find the equivalent resistance of the divider, and, if the total current entering the divider is 120 mA, find the current that flows in the 20 kΩ resistor.



2.F

(a) What values of voltage v and current i will be registered on the voltmeter and the ammeter shown in the circuit below?



(b) It is proposed to use d'Arsonval meters to make the measurements of voltage and current illustrated in the circuit above. Each meter has an internal resistance of 6Ω and full-scale current of 1mA. The voltmeter should read up to ±10V and the ammeter should read up to 5A.

What additional resistors are needed in each meter, and how should they be connected?

Chapter 3

Nodal and Mesh Analysis

In Lectures C4 and C5 we cover:

- Nodal Analysis - a circuit analysis technique that builds on KCL
- How to handle voltage sources in nodal analysis
- Mesh Analysis - another circuit analysis method, this time using KVL
- How to handle current sources in mesh analysis
- The concept of *duality*

3.1 Nodal Analysis

In Nodal Analysis, the unknowns are generally *voltages*. We choose to find voltages at the nodes of the circuit. Note that since voltages are always measured *across* an element, this means finding the voltages at each node, relative to some *reference node*, which we refer to as *zero potential*, or *earth* or *ground*. We generally choose the node in the circuit which has the most elements attached to it to be the ground.

So, if a circuit has N nodes, there will be $N - 1$ node voltages.

We get the equations we are looking for by applying KCL at the nodes. Remember that every current out of a node depends on the elements in its branch of the circuit, and on the voltage across them (use Ohm's Law). Also remember that the sum of the currents out of the node must be *zero*.

We arrive at one *node equation* per node (except for the ground), and should

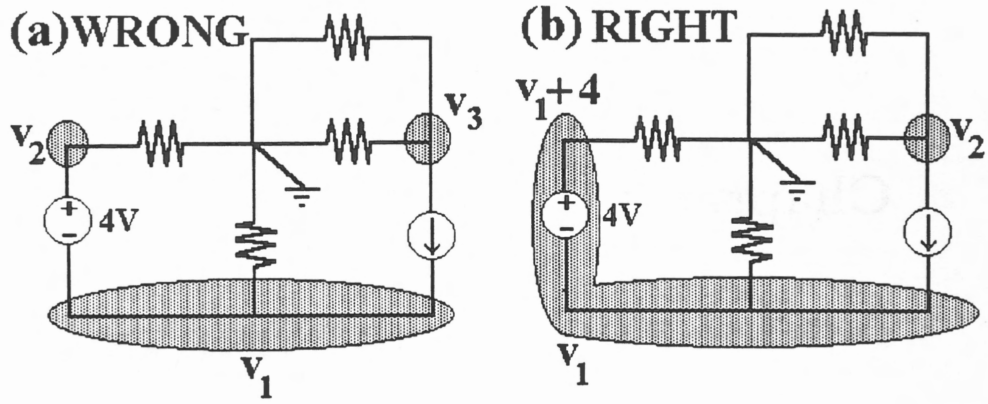


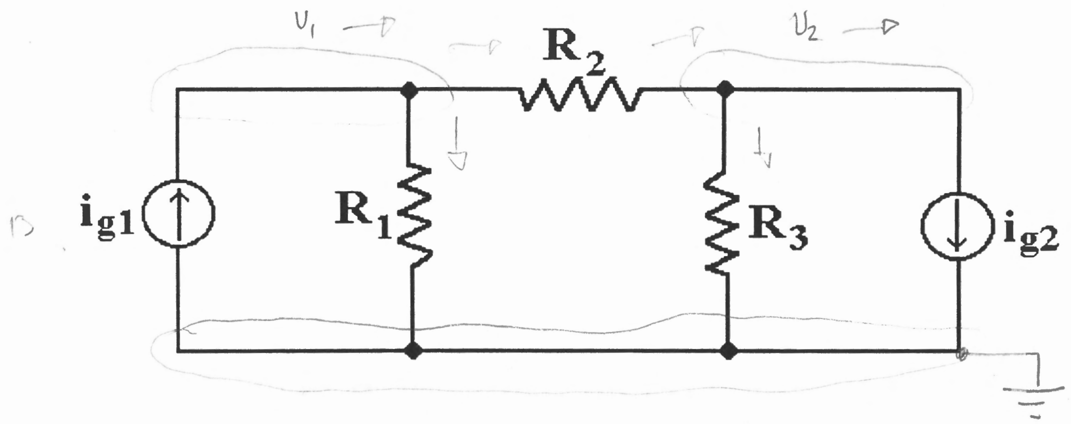
Figure 3.1: When and Why a Supernode is Necessary

thus be able to get $N-1$ simultaneous node equations in an N-node circuit. Each equation is a relationship between voltages, and if we solve simultaneously, we will get the voltages at all the nodes.

Once all of the node voltages are known, we can solve for unknown currents simply by using Ohm's Law. See Examples 3.1 and 3.2.

3.2 Nodal Analysis with Supernodes

Note that if you have a voltage source present, then solving the circuit will be easy if you make use of a *supernode*. You must use a supernode, because with ordinary nodes you will probably not know the current flowing through the voltage source. As an example, study Figure 3.1. At first sight you might choose nodes as in Figure 3.1(a), but you then cannot write a KCL equation for v_1, so the method will not work. Instead, use the arrangement in Figure 3.1(b). Note that in a supernode you label all of the internal voltages, using your knowledge of the size of the included voltage source (e.g. use labels v_1 and $v_1 + 4$ in this case). See Examples 3.3 and 3.4.



Example 3.1: Analyse this circuit using nodal analysis.
(i_{g1} = 13A, i_{g2} = 7A, R_1 = 1Ω, R_2 = 2Ω, R_3 = 1Ω)

Step 1: Identify the nodes (there are 3 of them)

Step 2: Choose a ground node (the busiest node)

Step 3: Label the other 2 nodes (v_1 and v_2)

Step 4: Obtain 2 equations using KCL

$$-13 + \frac{V_1 - 0}{1} + \frac{V_1 - V_2}{2\Omega} = 0$$ Currents leaving node V_1

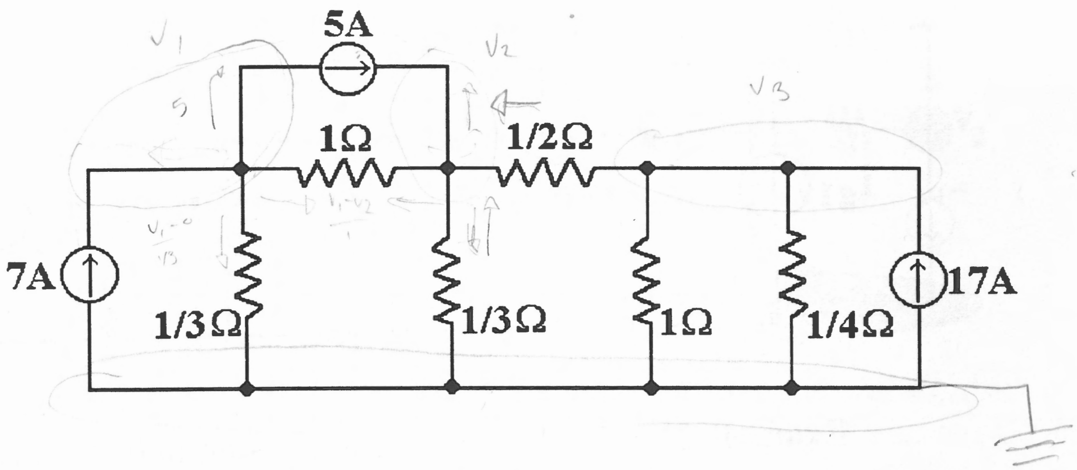
$$3V_1 - V_2 = 26$$

$$\frac{V_1 - V_2}{2\Omega} - 7A - \frac{V_2 - 0}{1\Omega} = 0 \qquad V_1 - 3V_2 = 14$$

Step 5: Simplify the equations

Step 6: Solve the equations (however you like)

$V_1 = 8V \qquad V_2 = -2V$



Example 3.2: Analyse this circuit using nodal analysis.

Step 1: Identify the 4 nodes

Step 2: Choose a ground

Step 3: Label v_1, v_2 and v_3

Step 4: Obtain 3 equations using KCL

$-7 + 3v_1 + \frac{v_1 - v_2}{1} + 5 = 0$

$\frac{v_2 - v_1}{1} - 5 + 2(v_2 - v_3) - \frac{v_2 - 0}{1/3\,\Omega}$

Step 5: Simplify

$4v_1 - v_2 = 2$

$-v_1 + 6v_2 - 2v_3 = 5$

$-2v_2 + 7v_3 = 17$

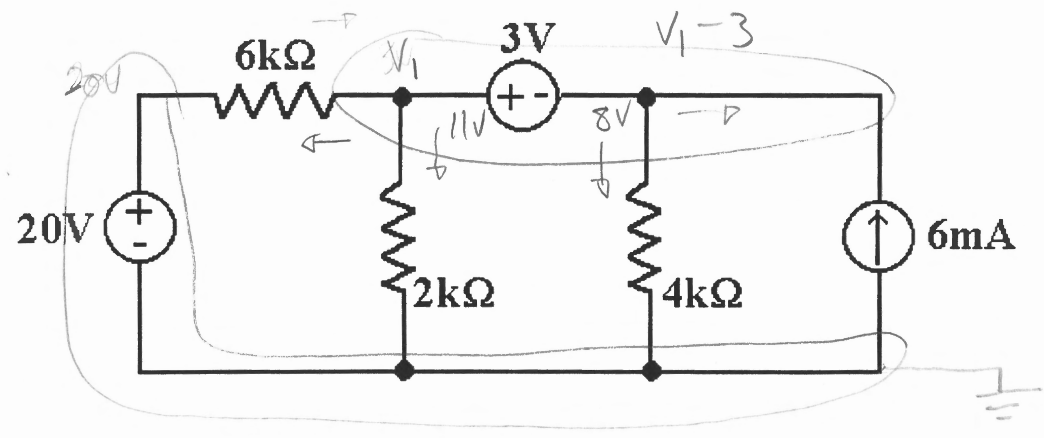
Step 6: Solve

$v_3 = 3V$ $v_2 = 2V$ $v_1 = 1V$

Note: we can now answer questions about current, power etc. For example:

(a) What current flows in the 1/2Ω resistor? 2A from right to left

(b) What power does the 5A current source deliver? 5W



Example 3.3: Analyse this circuit using nodal analysis.

Step 1: Identify the nodes and/or supernodes

Step 2: Choose the earth node

Step 3: Label the nodes and all parts of the supernodes

Step 4: Obtain the KCL equation (there is only 1)

Start at V_1 $\quad \frac{V_1-20}{6000} + \frac{V_1}{2000} + \frac{V_1-3}{4000} - 6mA = 0$

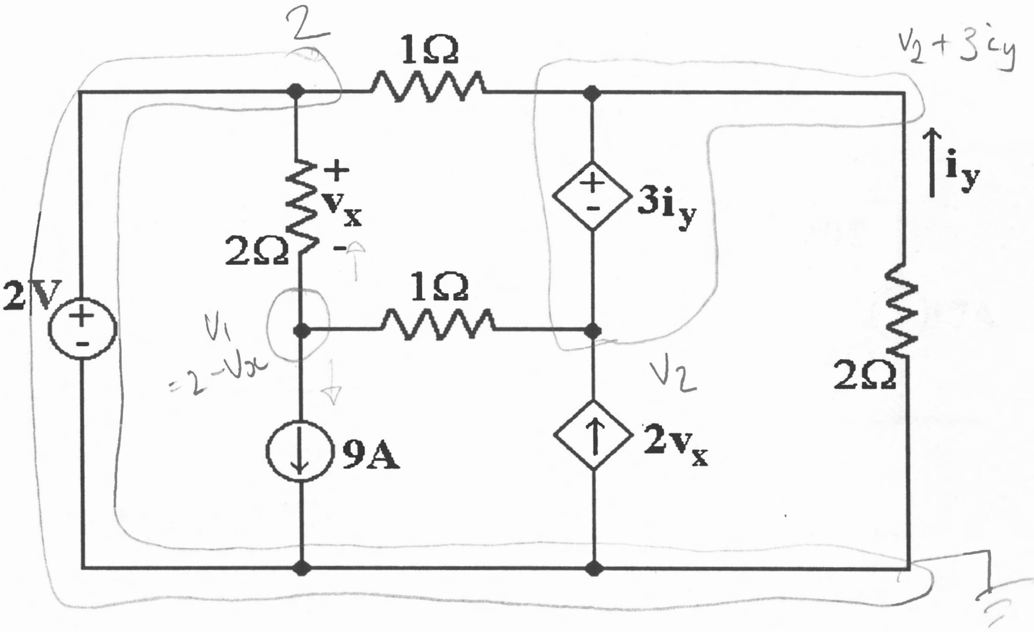
Step 5: Simplify it

$2V_1 - 40 + 6V_1 + 3V_1 - 9 - 72 = 0$

$121 = 11V_1 \qquad V_1 = 11$

Step 6: Solve and interpret your result

Question: What is the current in the 2kΩ resistor? $\frac{11V}{2k\Omega} = 5.5mA$



Example 3.4: Find v_x and i_y using nodal analysis.

$$9 + \frac{V_1 - 2}{2} + \frac{V_1 - V_2}{1\Omega} = 0$$

$$\frac{V_2 - V_1}{1} - 2(2 - V_1) + \frac{V_2 + 3_y - 2}{1}$$

Get $i_y = \frac{-V_6}{5}$

$V_1 = -2V$ $\quad V_x = 4V$

$V_2 = 5V$ $\quad i_y = -1A.$

3.3 Mesh Analysis

In Mesh Analysis, the unknowns are generally *currents*, and we choose to identify closed paths or *loops* in our circuit to find these unknown currents. We shall use the word *loop* to mean any closed path in a circuit (perhaps containing smaller loops within it), and the word *mesh* to mean a loop which contains no smaller loop within itself (i.e. a mesh contains no element within its closed path).

We can get the *mesh equations* of a circuit by applying KVL around as many independent loops as we can find. To ensure independence, we shall use no loop all of whose sub-loops or meshes have already been used.

So we arrive at as many mesh equations as there are meshes in the circuit. Each equation relates unknown currents, and there will always be enough meshes to find as many simultaneous equations as there are unknowns.

Once all of the loop currents are known, we can find actual currents in the circuit, or use Ohm's Law to find any particular voltage across an element of known resistance. See Examples 3.5 and 3.6.

If there is a current source present, then matters are actually easier. In effect, it means that you already know one of the mesh currents, so you don't need to have the mesh equation that it gave rise to. This makes a smaller and easier set of simultaneous equations to solve. See Example 3.7.

3.4 Duality

You will probably have noticed how many concepts in circuit analysis come in pairs. Think for example of *voltage* and *current*, or of *resistance* and *conductance*, giving rise to two forms of Ohm's Law, either $v = iR$ or $i = vG$. Other examples are *short circuit* (where $v = 0$) and *open circuit* (where $i = 0$), or else *KVL* and *KCL*. Most recently we have encountered *node analysis* and *mesh analysis*.

Paired concepts of this sort are called *duals*, and the existence of duality in circuit analysis hints at an underlying unity in all of the theories that we study.

An interesting spin-off from duality is that, given a simple circuit, you can find its dual by replacing all of its elements by their respective duals. All

the voltages and currents in the original circuit are then equal to their duals in the dual circuit. This remarkable fact is of value in more advanced circuit analysis.

The diagram below shows a circuit consisting of resistors and a current and

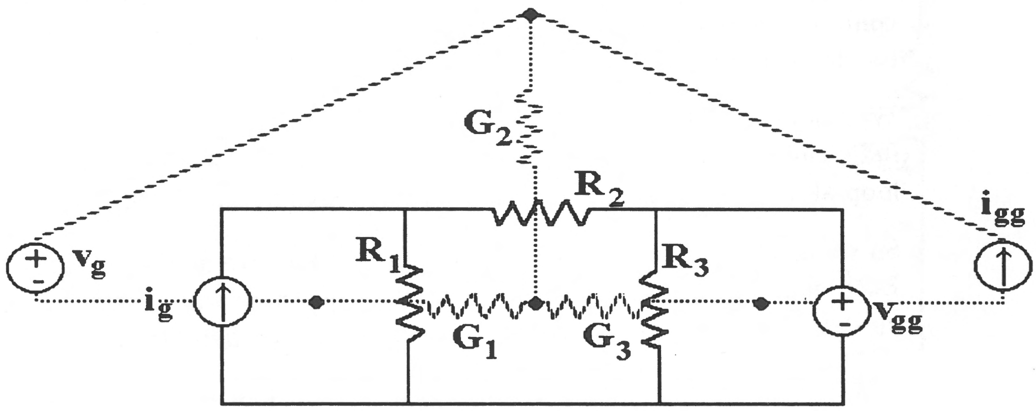
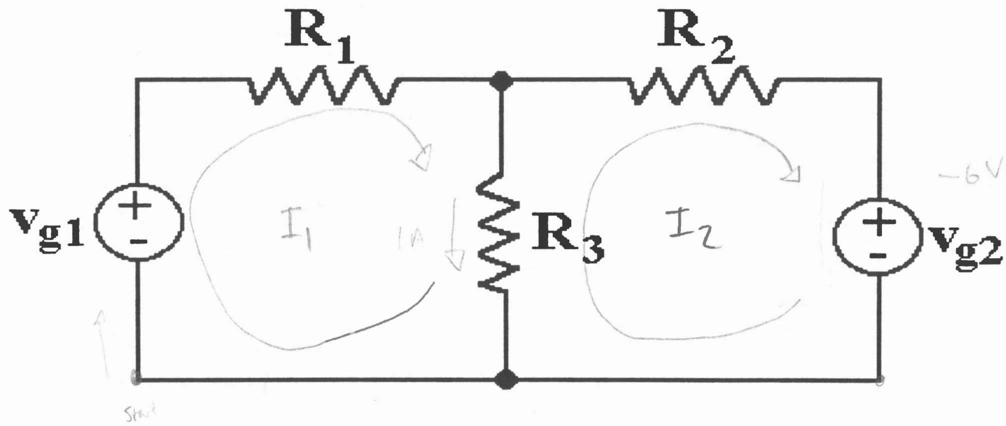


Figure 3.2: A resistive circuit, and its dual (shown dotted)

voltage source, and its dual, consisting of all dual elements. You will notice that the dual of a mesh-centre is a node, and vice-versa.



Example 3.5: Analyse this circuit using mesh analysis.
($v_{g1} = 7\text{V}$, $v_{g2} = -6\text{V}$, $R_1 = 1\Omega$, $R_2 = 2\Omega$, $R_3 = 2\Omega$)

Step 1: Identify all the mesh currents (it doesn't matter how they go in)

Step 2: Note that the mesh currents are not necessarily real currents which actually flow in the circuit. Label the real currents, using capital letters

Step 3: Use KVL to get the mesh equations (there are 2 of them this time)

$7V - I_1 1\Omega - (I_1 - I_2)2 = 0$ $\quad 3I_1 - 2I_2 = 7$ (1)

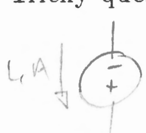
$-2(I_2 - I_1) - 2I_2 + 6V = 0$ $\quad 2I_1 - 4I_2 = -6$

Step 4: Solve the equations however you like

$I_1 = 5A$

$I_2 = 4A$

Tricky question: How much power does v_{g2} generate?

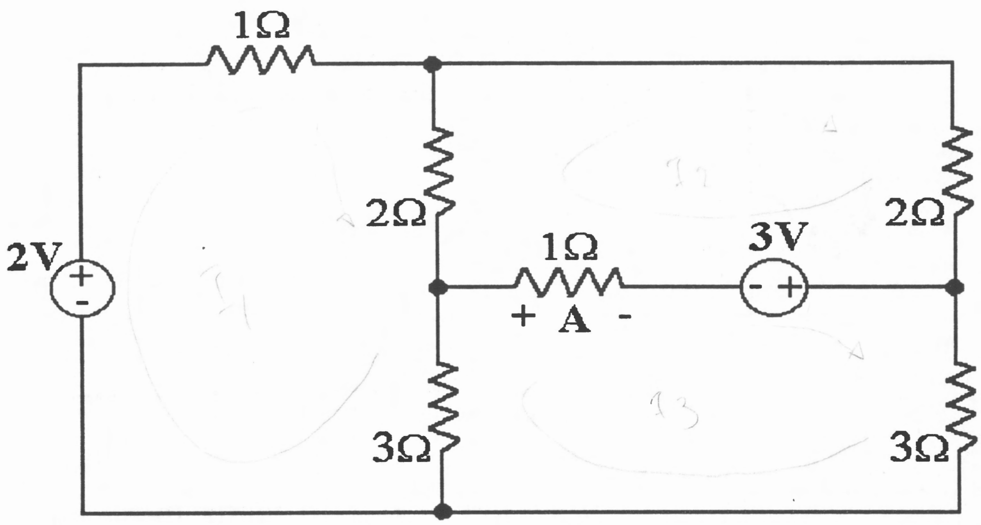


$P = UI = 6 \times 4$

$= 24W$ delivered

1A $\quad i^2R$

$V = 2V \quad P = i^2R = 2 \times 1 = 2W$



Example 3.6: Analyse this circuit using mesh analysis.

Step 1: Identify all the mesh currents

Step 2: Use KVL to get the 3 mesh equations

$+2V - I_1 - (I_1 - I_2)2 - 3(I_1 - I_3) = 0$

$-3V - 1(I_2 - I_3) - 2(I_2 - I_1) - 2I_2 = 0$

$-3(I_3 - I_1) - 1(I_3 - I_2) + 3 - 3I_3 = 0$

Step 3: Simplify

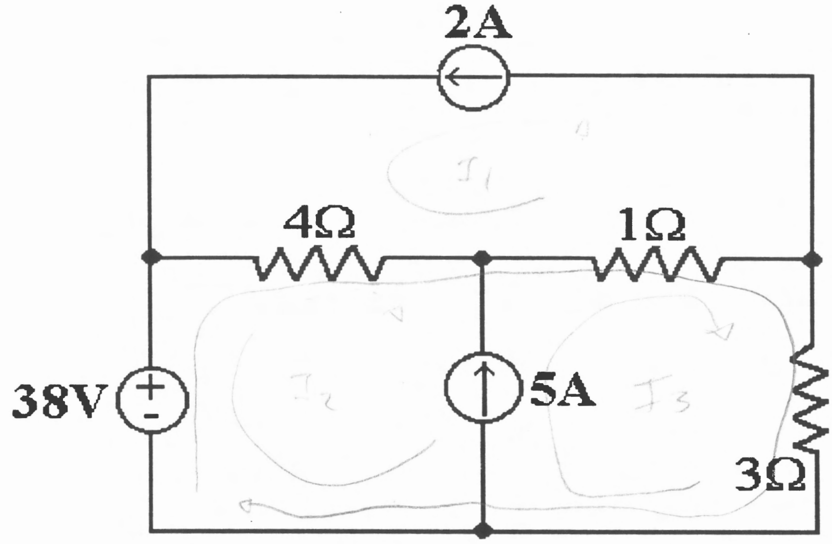
Step 4: Solve the equations however you like

$I_1 = -571.4\,mA$

$I_2 = -815.1\,mA$

$I_3 = 67.23\,mA$

Can you deduce from these calculations that the voltage across the resistor marked A is 0.883V, with the polarity as marked? YES



$I_1 = -2A$

$I_3 - I_2 = 5A$

Example 3.7: Analyse this circuit using mesh analysis.
(Because there are two current sources, we can write two equations for the mesh currents, and are left with only one loop to which to apply KVL)

Choose a loop which does not contain a current source

$38V - 4(I_2 + 2A) - 1(I_3 + 2A) - 3I_3 = 0$

$I_2 + I_3 = 7A$

$-I_2 + I_3 = 5$

$2I_3 = 12A$

$I_3 = 6A \quad I_1 = -2A$

$I_2 = 1A$

Find the voltage across the 5A current source, and hence calculate the power that it generates.

$38 - 4(1A + 2A) + V_{current} = 0$

$38 - 12 = V_{current}$

$V_{current} = 26V$

TUTORIAL 3

3.1 Using nodal analysis, find v_1 and v_2 in Figure 3.3.
(64V,48V)

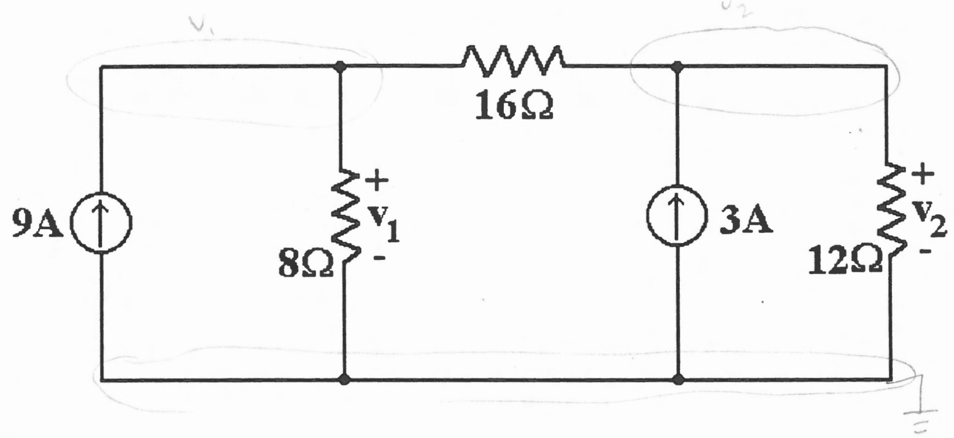


Figure 3.3: Figure for Question 3.1

3.2
Using nodal analysis, find v_x, v_y and v_z in Figure 3.4.
(-20V, -120V, 100V)

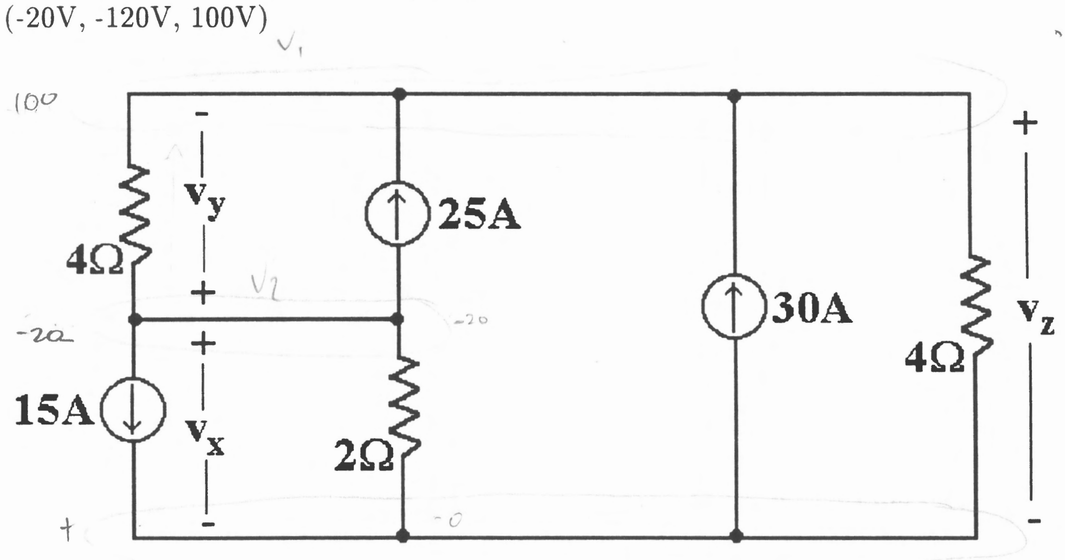


Figure 3.4: Figure for Question 3.2

3.3
Using nodal analysis, find the current i_x in Figure 3.5.
(4A)

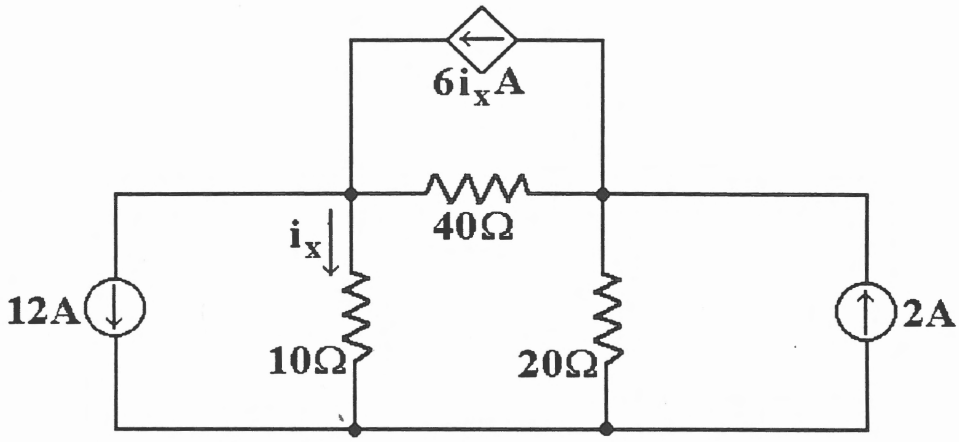


Figure 3.5: Figure for Question 3.3

3.4
Using nodal analysis, find the current i_x in Figure 3.6.
(3.5A)

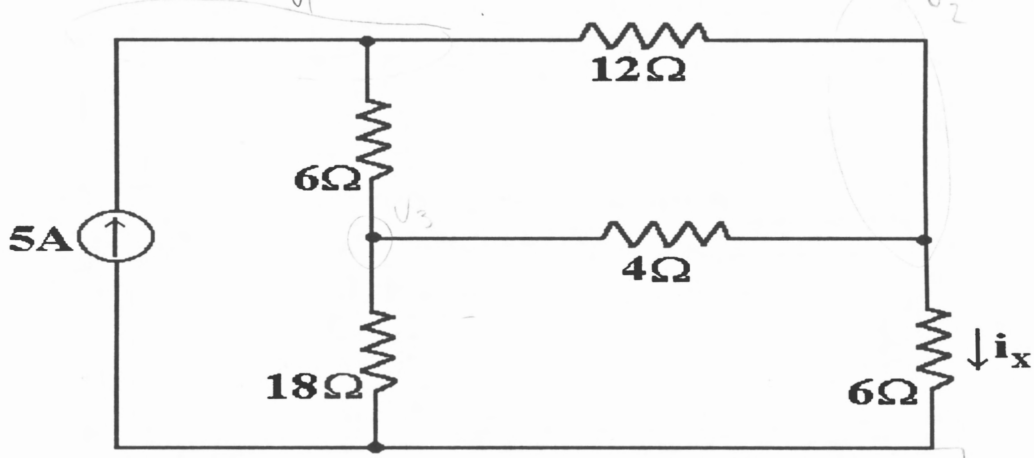


Figure 3.6: Figure for Question 3.4

3.5
Using nodal analysis, find the voltage v_x in Figure 3.7.
(20V)

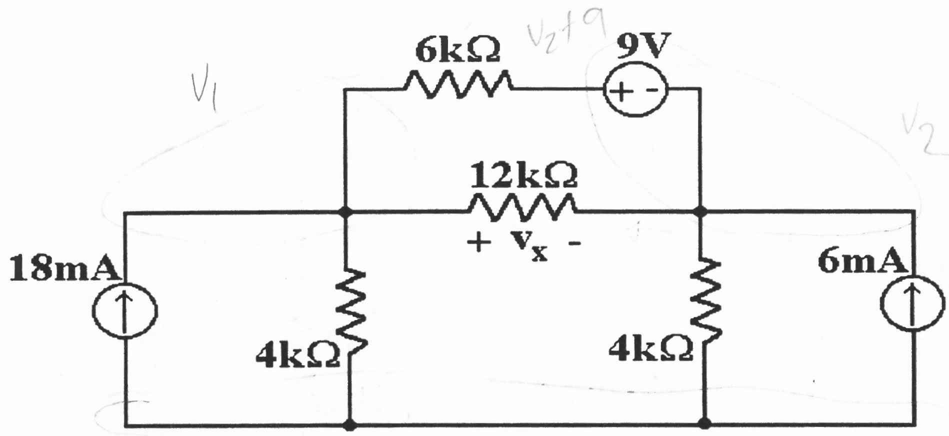


Figure 3.7: Figure for Question 3.5

3.6
Using mesh analysis, find i_x and i_y in Figure 3.8 if the element marked X is a 6V independent voltage source with its positive terminal at the top.
(2A, 1A)

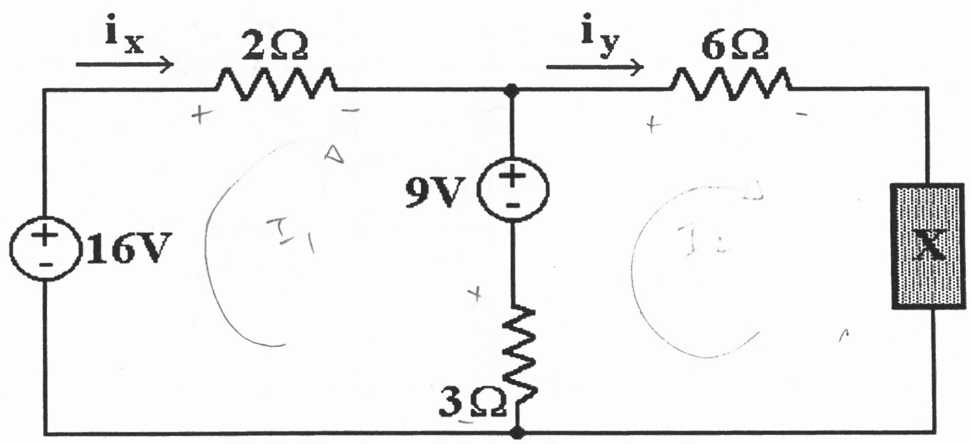


Figure 3.8: Figure for Questions 3.6 and 3.7

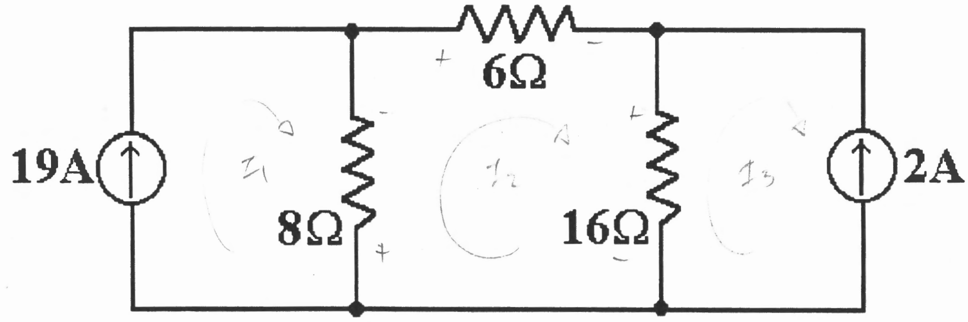


Figure 3.9: Figure for Question 3.8

3.7
Using mesh analysis, find i_x and i_y in Figure 3.8 if the element marked X is a dependent voltage source of $6i_x$V, with its positive terminal at the bottom.
(5A, 6A)

3.8
Use mesh analysis in Figure 3.9 to find the current flowing through and the power dissipated by:
(a) the 16Ω resistor
(b) the 6Ω resistor
(6A, 576W; 4A, 96W)

3.9
In Figure 3.10, find the voltage v_x using either method.
(26V)

3.10
Use mesh analysis to find the value of v_x in Figure 3.11.
(5V)

$6i_y - 6i_x - 3i_2$

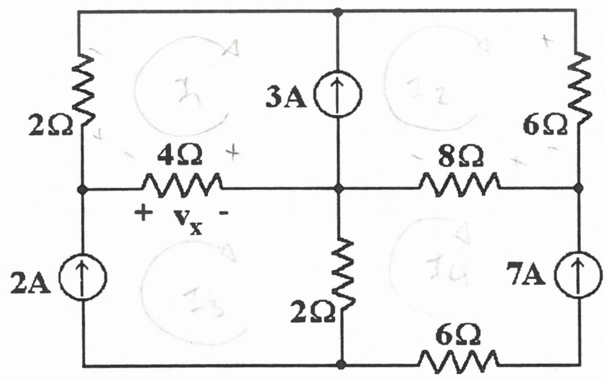


Figure 3.10: Figure for Question 3.9

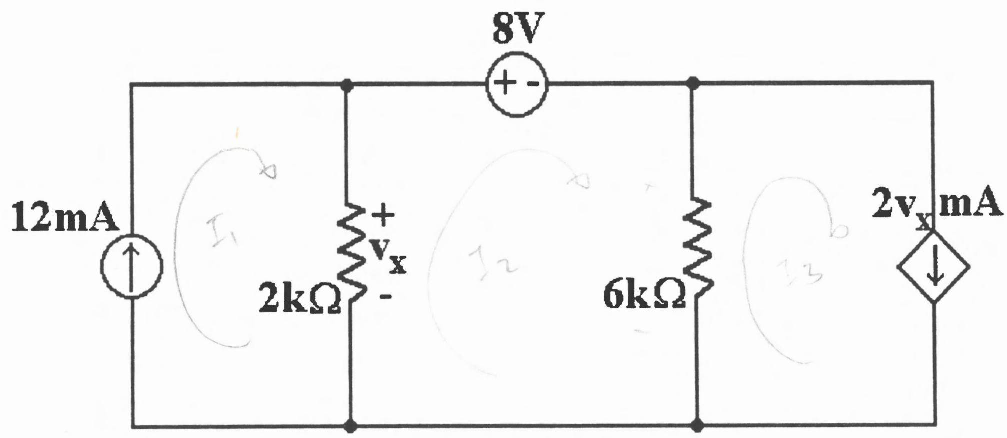
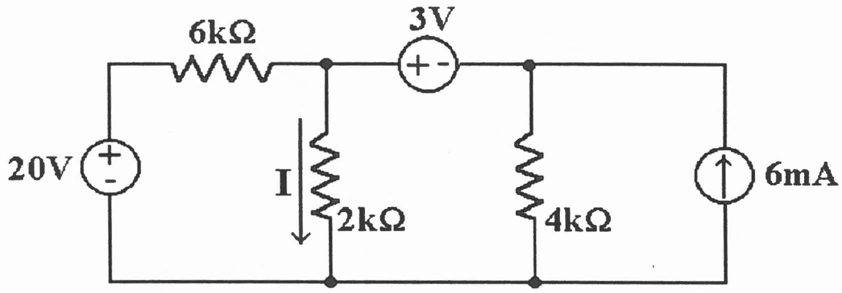


Figure 3.11: Figure for Question 3.10

WORKSHEET 3

3.A

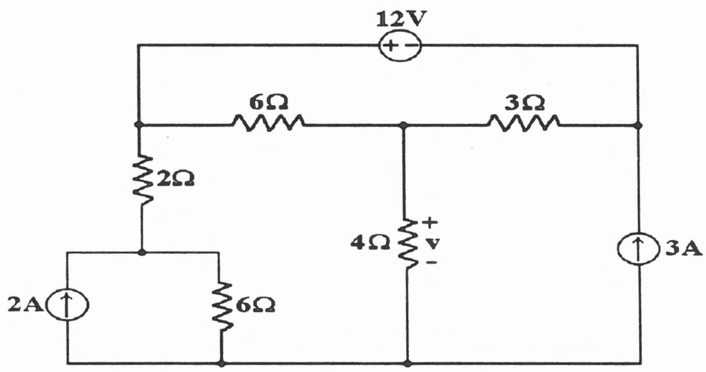
Use *nodal analysis* to find the current I that flows in the 2kΩ resistor in the



circuit above. Then confirm your answer by using *mesh analysis.* Show your working carefully in each case.

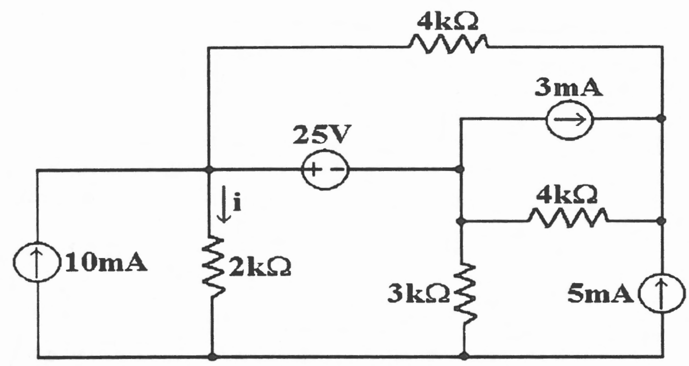
3.B*

In the circuit below, the voltage v can be found using either nodal or mesh analysis. **Explain** which method would require a smaller set of simultaneous equations, and then use the simpler method to find v.



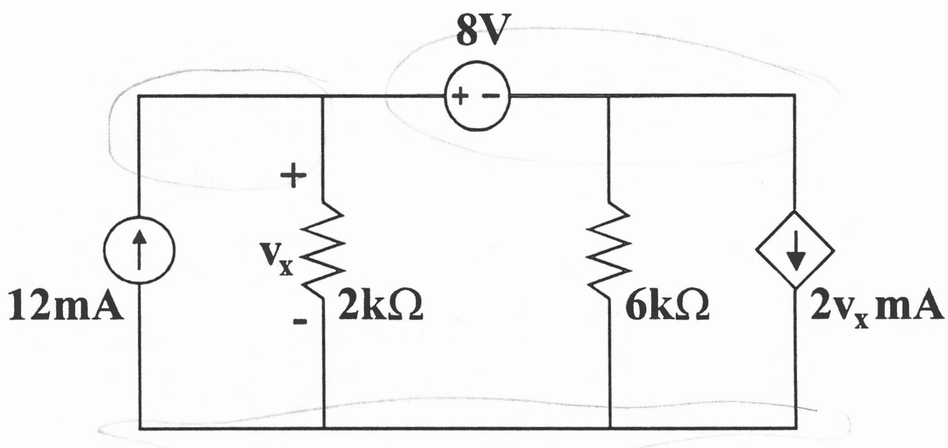
3.C

Use nodal *or* mesh analysis to find current i in the circuit below.



3.D*

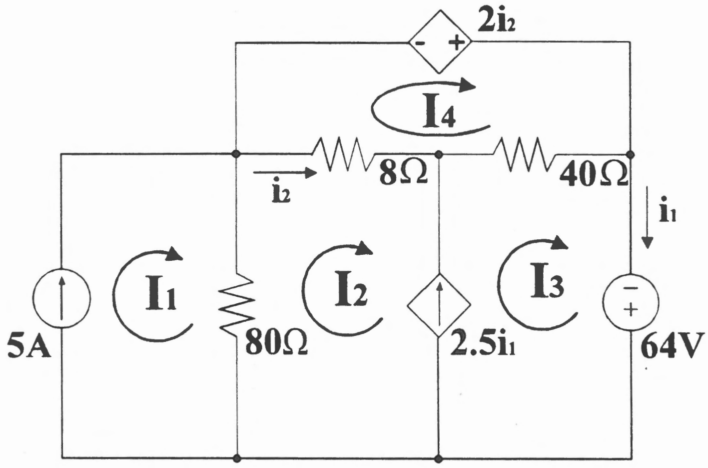
Use *nodal analysis* to find the value of V_x in the circuit above. Then confirm



your answer by using *mesh analysis*. Show your working carefully in each case.

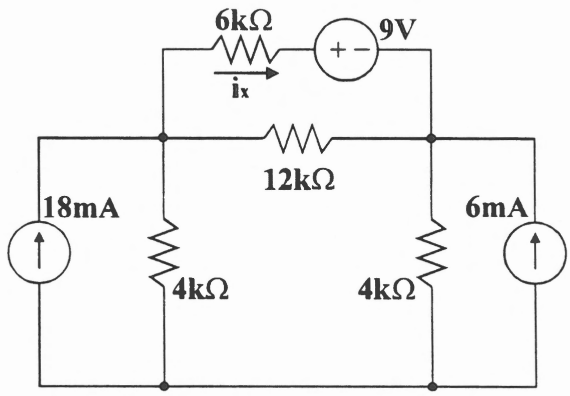
3.E

Use *mesh* analysis, with the mesh currents I_1, I_2, I_3, and I_4 as given in the circuit below, to find the currents i_1 and i_2.



3.F

Use *nodal analysis* to solve the circuit below, and hence find the current i_x. Show all of your working carefully.



Chapter 4

Further Network Theorems

In Lectures C6 and C7 we cover:

- The so-called *Proportionality Property* of linear circuits
- Solving *ladder networks* using proportionality
- A fact about linear circuits called the *Superposition Theorem*
- The Theorems of Thévenin and Norton
- Transfer of maximum power from source to load

4.1 The Proportionality of Linear Circuits

We can think of a resistor as having an *input* and an *output*. If the input is a current, then the output is a voltage given by $v = Ri$. If, however, the input is a voltage, then the output is the current given by $i = Gv$. Either way, since $R = 1/G$ is a constant, we can see that if we choose to multiply the input by some factor, the output would also be multiplied by the same factor.

This feature is known as the *proportionality property*, and any circuit element for which it holds is called a *linear element*. A circuit containing only linear elements and perhaps some independent sources is termed a *linear circuit*. Its input is the sum of all the sources, and its output is the voltage across or the current through any of its passive elements.

We can get equations which describe such a circuit from KVL or KCL, and they all typically look like this:

$$a_1v_1 + a_2v_2 + \ldots + a_nv_n = f \text{ (from KVL)}$$
$$\text{or } a_1i_1 + a_2i_2 + \ldots + a_ni_n = g \text{ (from KCL)}$$

where f is the sum of all the independent voltage sources in some loop, and g is the sum of all the independent current sources near some node. The a_i are all 0 or ± 1. It is clear from these equations that if we multiply the input (f or g) by a factor, all of the outputs are multiplied by the same factor. Hence, the proportionality theory applies to linear circuits.

There are two important consequences of this:

1. In general, we may use all of the tools of *linear algebra* (MAM280W) to help us to analyse circuits
2. In particular, if in any circuit we change all of the sources by a factor (e.g. double them), then any given current or voltage in the circuit will also change by that factor (i.e. it will double too).

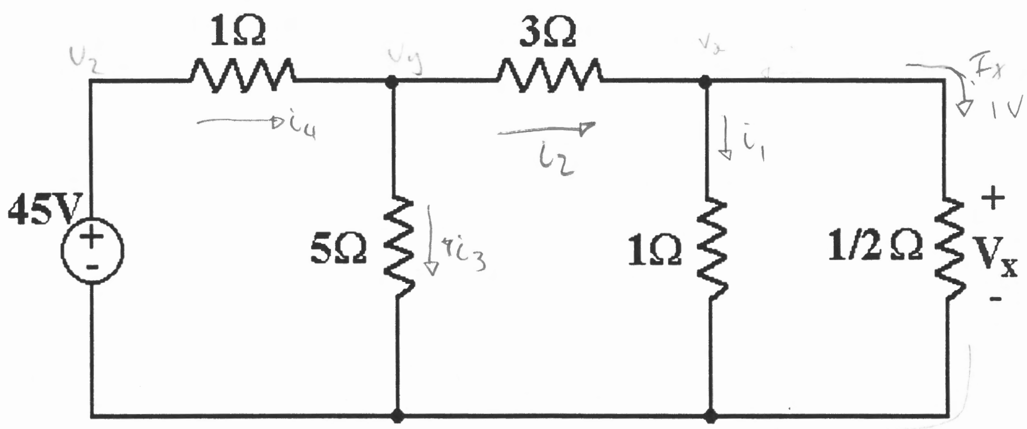
The proportionality property allows us to judge the effect of changing the sources much more quickly than a complete re-analysis of the circuit.

4.2 Solving Ladder Networks

A useful application of proportionality is in solving long strings of parallel and serial elements called *ladder networks*. The technique is to assume the unknown voltage or current to have unity value, and then to work back from there to the source using simple circuit laws. The result is compared to the true value of the source, and the assumed initial value is finally adjusted proportionally to give the true value of the unknown quantity.

An example of how this is done is given below, where the aim is to find the voltage v_x.

Example 4.1: Find v_x



Step 1: Label all of the voltages and currents in the ladder.
Step 2: Assume the unknown (voltage v_x in this case) to be equal to 1.
Step 3: Work back to the source

If $v_x = 1$, $i_x = \frac{v_x}{1/2} = 2A$ $\quad i_1 = \frac{v_x}{1} = 1A$

∴ by KCL, $i_2 = i_1 + i_x = 1 + 2 = 3$

Then $v_y = v_x + 3i_2 = 1 + 3(3) = 10V$

$i_3 = \frac{10V}{5\Omega} = 2A$

∴ by KCL $i_4 = i_3 + i_2 = 2 + 3 = 5$

Hence, $V_z = v_y + i_4 = 10 + 5 = 15V$

Step 4: The actual source is 45V, so adjust v_x proportionally.

$1 = 15$

$3 = 45$

$V_x = 3V$, by principle of proportionality

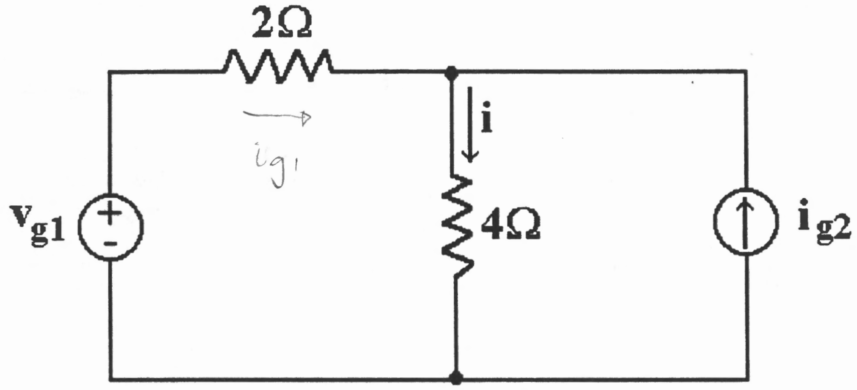
Watch out for DIODES, or dependent source, or TRANSISTOR

4.3 The Principle of Superposition

Let us find the current i in the two-source circuit below. By KCL, $i_{2\Omega} = i - i_{g_2}$, so by KVL in the left mesh $v_{g_1} - 2(i - i_{g_2}) - 4i = 0$, or $v_{g_1} + 2i_{g_2} = 6i$. Therefore:

$$i = v_{g_1}/6 + i_{g_2}/3$$

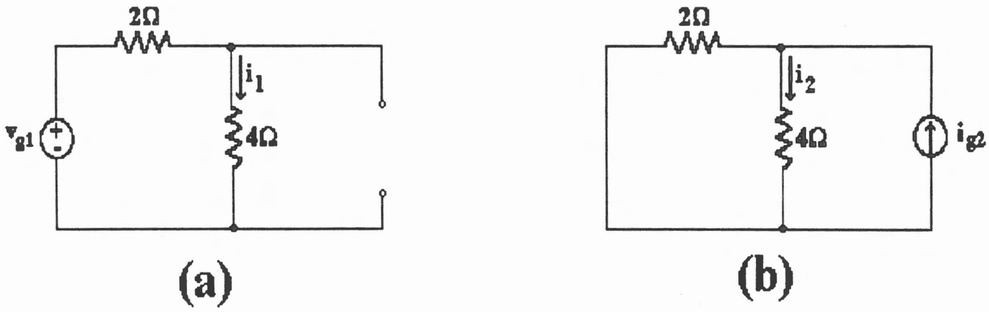
Note that there is a contribution to i from each of the two independent sources. This suggests that we might be able to evaluate each contribution separately, by considering each source *one at a time*. It turns out that in any *linear circuit*, this approach is valid for all *independent sources*. We can



test this theory in the above circuit. First, set the current source to zero (i.e. replace it with an open circuit as in Figure (a) below). Clearly $i_1 = v_{g_1}/6$. Now set the voltage source to zero (i.e. replace it with a short circuit as in Figure (b)), and by current division, we see that $i_2 = \frac{2}{2+4} i_{g_2} = i_{g_2}/3$. Then, by the superposition theorem, we re-obtain $i = i_1 + i_2 = v_{g_1}/6 + i_{g_2}/3$.
We can use the superposition theorem to make calculations of this sort much easier in circuits where there are several independent sources, but we should note:

- The theorem does *not* apply if the circuit is not linear (i.e if it contains any non-linear elements)

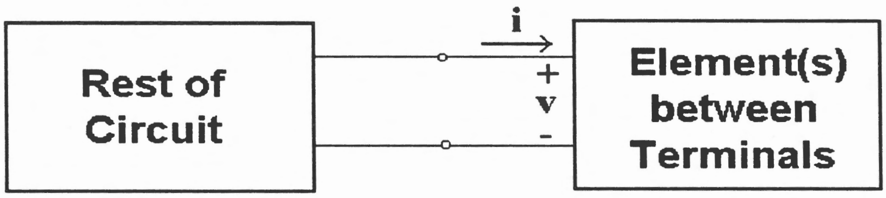
- The method applies to calculations of voltage or of current, but *not* of power. Power dissipation in an element must be calculated indirectly from voltages and/or currents



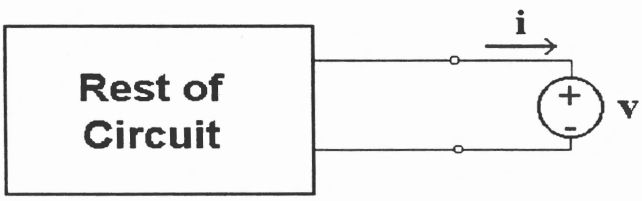
- Dependent sources are *not* set to zero, and make the method messy. We do not generally use superposition where there are many dependent sources in the circuit, although it can be made to work.

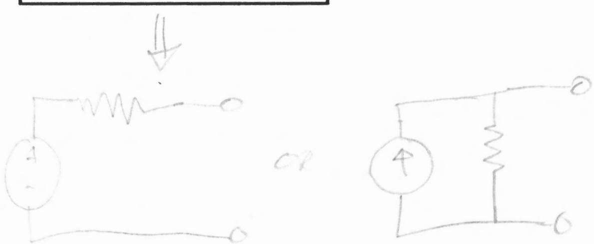
4.4 Thévenin and Norton's Theorems

These theorems allow entire circuits, viewed from a given pair of terminals, to be replaced by a very simple equivalent consisting of a single resistor and a source. The theorems are useful for calculating the voltage or current of a single element, because with the rest of the circuit replaced by its simple equivalent, analysis of the whole circuit becomes very easy.



is equivalent to

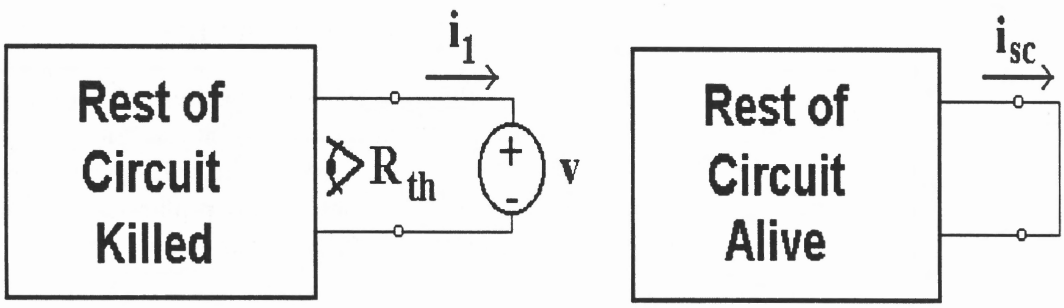




The choice of two terminals to "look into" actually splits the circuit into two parts (i.e. the element(s) between the terminals chosen, and the rest of the circuit). As far as the rest of the circuit is concerned, the element(s) between the terminals can be replaced by anything with the same voltage across it which accepts the same current - an ideal voltage source, for example.

The current i may now be found by superposition:

- "kill" all of the independent sources in the rest of the circuit to get current i_1 flowing into the voltage source
- "kill" the voltage source (i.e. short-circuit it) to get i_{sc} flowing out of the rest of the circuit
- Then $i = i_1 + i_{sc}$

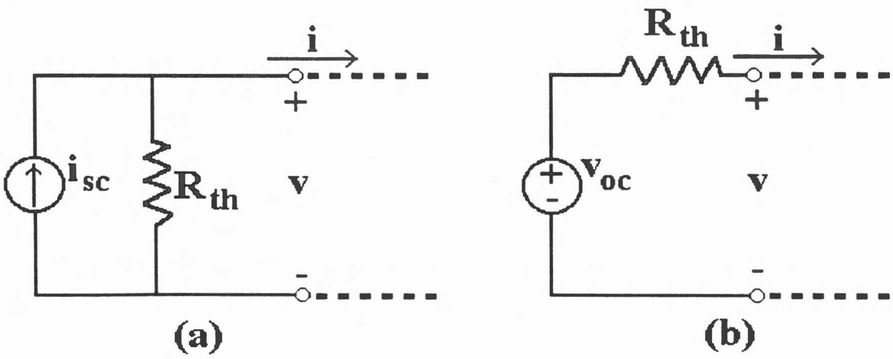


While the rest of the circuit is killed (its voltage sources shorted and its current sources open-circuited), the voltage source sees only resistors, and we could use series and parallel combination laws to work out the equivalent resistance, which is denoted R_{th}. Then, by Ohm's law, $i_1 = -v/R_{th} = i - i_{sc}$, so

$$\boxed{i = -v/R_{th} + i_{sc}}$$

This formula applies for any voltage, so we might choose to make v equal to the open circuit voltage v_{oc} (in which case i must be zero). This gives $0 = -v_{oc}/R_{th} + i_{sc}$, and hence $i_{sc} = v_{oc}/R_{th}$. The boxed equation above now gives us $i = (-v + v_{oc})/R_{th}$, and so

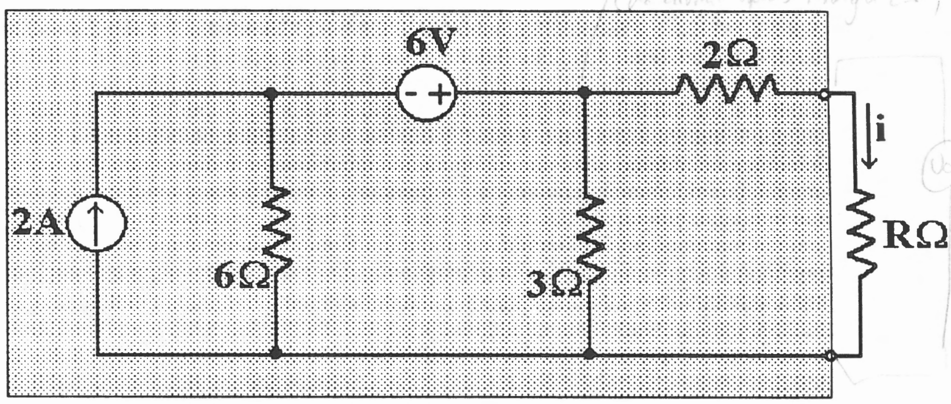
$$\boxed{v = -iR_{th} + v_{oc}}$$

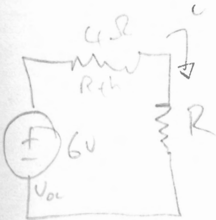


Now the first boxed equation clearly describes Figure (a) above (use KCL). This is known as the *Norton equivalent circuit.* It replaces the rest of the original circuit with a current source equal to the short-circuit current, in parallel with the Thévenin resistance. Meanwhile, the second boxed equation is achieved by the circuit of Figure (b) (use KVL). This circuit is known as the *Thévenin equivalent circuit.* It replaces the rest of the original circuit with a voltage source equal to the open circuit voltage, in series with the Thévenin resistance.

Example 4.2: Find the Thévenin and Norton Equivalents of this circuit, and use them to find expressions for the current i flowing through resistor R

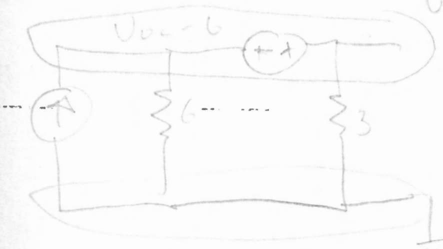
V_{oc} (no current flows through 2Ω, RΩ is shorted out)





$i = \frac{6}{4 + V_{oc}}$

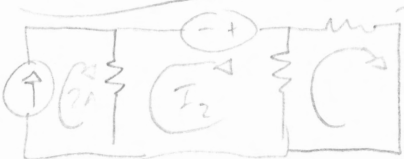
1. Thevenin equivalent:
Short out 6V, open 2A, $R_{th} = 4\Omega$
Find V_{oc} at terminals. Use supernodes



$-2 + \frac{V_{oc} - 6}{6} + \frac{V_{oc}}{3} = 0$

$\therefore V_{oc} = 6V$

Norton equivalent

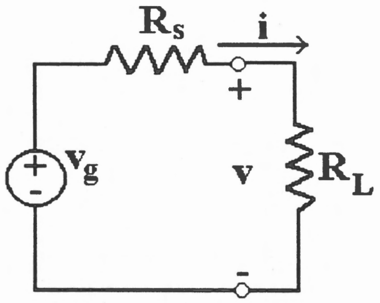


i_{sc} use mesh analysis

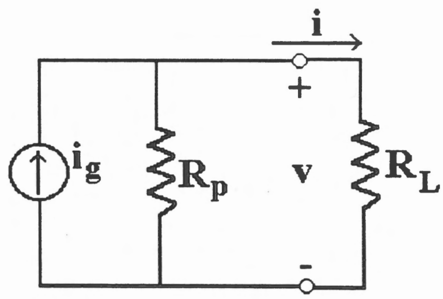
$i_{sc} = 1.5A$

4.5 Maximum Power Transfer

A *real* source does not behave ideally, because it contains resistance which prevents it from continuing to supply steady voltage or current regardless of the load. We model this internal resistance as a series resistance R_s for real voltage sources, and as a parallel resistance R_p for real current sources. If we now connect a load R_L to such real sources, we will have one of the two circuits shown below.



A Real Voltage Source Driving a Load



A Real Current Source Driving a Load

In the first case, the current $i = \frac{v_g}{R_s+R_L}$, so the power being delivered to the load by the voltage source is

$$P_L = i^2 R_L = \left[\frac{v_g}{R_s+R_L}\right]^2 R_L.$$

In the second case, the voltage across the load is $v = \frac{R_P R_L}{R_P+R_L} i_g$, so the power delivered to the load by the current source is

$$P_L = v^2/R_L = \left[\frac{R_P i_g}{R_P+R_L}\right]^2 R_L.$$

In many circuit applications, people are very interested in making sure that the power delivered to the load is maximised, so as to prevent wastage and inefficiency. The question arises as to what size R_L should be to achieve this aim. The answer requires some simple calculus. For maximum power transfer from the real voltage source:

$$\frac{dP_L}{dR_L} = v_g^2 \left[\frac{(R_s+R_L)^2 - 2R_L(R_s+R_L}{(R_s+R_L)^4}\right] = 0$$

From this $\quad (R_s + R_L)^2 = 2R_L(R_s + R_L)$

so $\quad R_s + R_L = 2R_L$

and so $\quad R_s = R_L$

You should now find it easy to prove that the maximum power transfer from the real current source occurs when $R_p = R_L$.

Furthermore, by re-substitution into the power expressions, it is found that the maximum power transfer is

$$P_{MAX} = \frac{v_g^2}{4R_s} \text{ for the voltage source,}$$

and

$$P_{MAX} = \frac{i_g^2 R_p}{4} \text{ for the current source.}$$

You should be aware of how to prove all of these relations.

From all of this, two things should be clear:

- If the load resistance equals the source's internal resistance, then the power output from the source is $v_g^2/2R_L$, half of which is dissipated in the internal resistor and half in the load
- *A quick way to achieve maximum power transfer from any circuit is to load it with its Thévenin equivalent resistance, because the above equations will then hold* by definition.

TUTORIAL 4

WORKSHEET 4

4.1

Find v and i in Figure 4.1, using the principle of proportionality.
(4V, 3A)

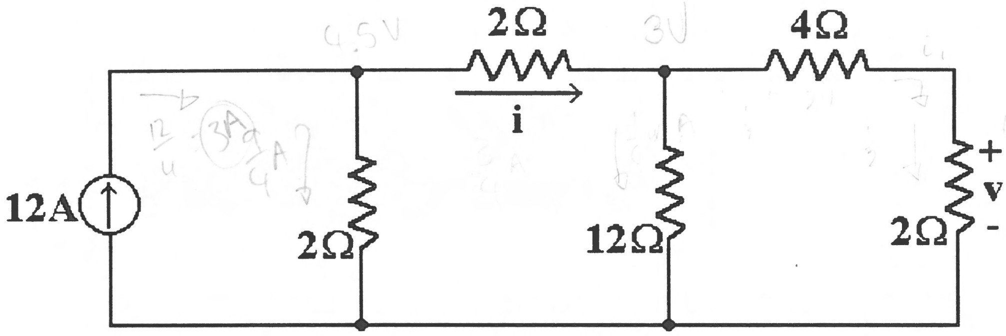


Figure 4.1: Figure for Question 4.1

4.2

A circuit is made of a voltage source v_g, a 2Ω linear resistor, and a nonlinear resistor all in series. The nonlinear resistor is described by $v = i^2$, where v is the voltage across the resistor and i, which is constrained to be non-negative, is the current flowing into the positive terminal. Find the current flowing out of the positive terminal of the source if:

(a) $v_g = 8\text{V}$

(b) $v_g = 16\text{V}$

Note that the proportionality property does not apply.

(2A, 3.123A)

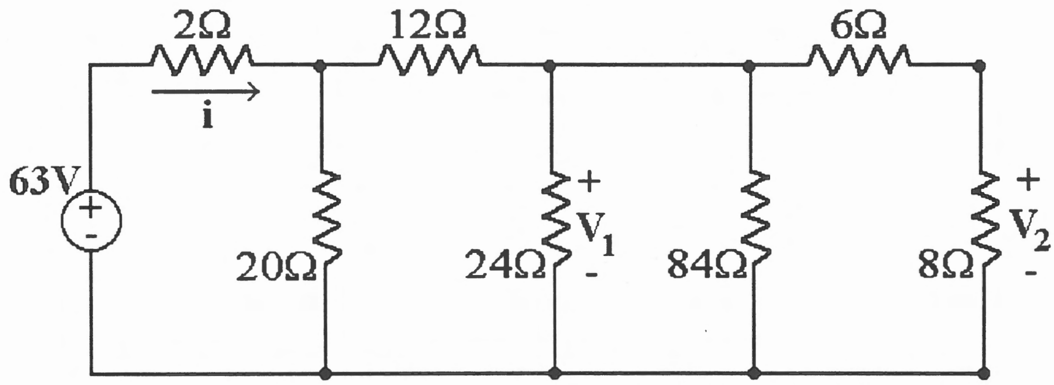


Figure 4.2: Figure for Question 4.3

4.3
Use the principle of proportionality to find i, v_1 and v_2 with reference to Figure 4.2.
(5.25A, 21V, 12V)

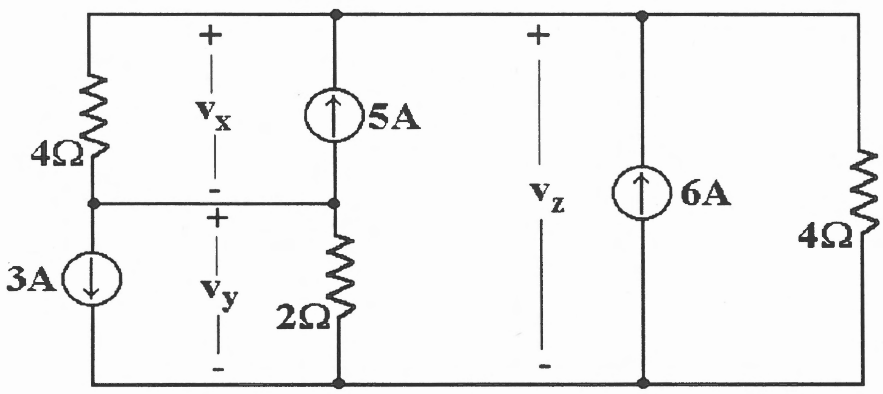


Figure 4.3: Figure for Question 4.4

4.4
Use the principle of superposition to find v_x, v_y and v_z in Figure 4.3.
(24V, -4V, 20V)

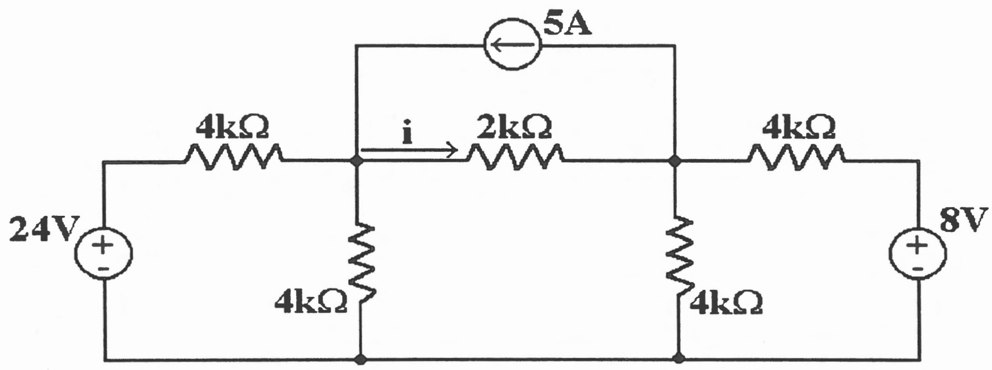


Figure 4.4: Figure for Question 4.5

4.5
By superposition, find i to 4 decimal places in Figure 4.4.
(3.3347A)

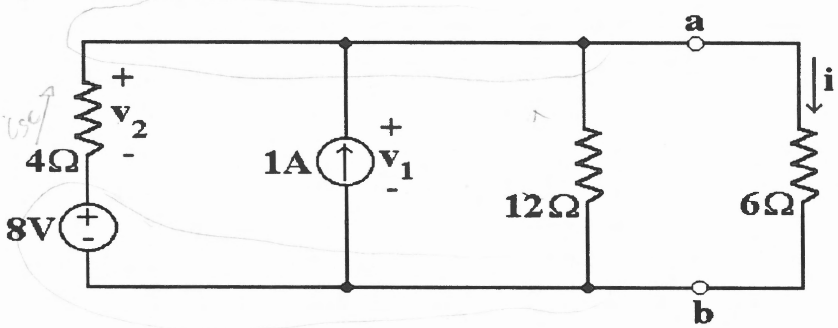


Figure 4.5: Figure for Question 4.6

4.6
With reference to Figure 4.5:
(a) Replace the network to the left of terminals a:b by its Thévenin equivalent circuit, and use the result to find i.
(b) Replace everything in the circuit except the 1A source by its Thévenin equivalent circuit and use the result to find v_1.
(c) Replace everything in the circuit except the 4Ω resistor by its Norton equivalent circuit and use the result to find v_2.
(3Ω, V_{OC} = 9V, 1A; 2Ω, V_{OC} = 4V, 6V; 4Ω, $I_{SC} = -1$A, -2V)

4.7

(a) Find i by replacing the network to the left of terminals a:b in Figure 4.6 by its Norton equivalent.

(b) If the dependent current source in Figure 4.6 is exchanged for an 11A independent current source acting in the same direction (left to right), find v by replacing the network to the right of terminals c:d by its Thévenin equivalent.

($i_{sc} = 3\text{A}$, $R_{th} = 8\Omega$, $i = 1\text{A}$; $v_{oc} = -4\text{V}$, $R_{th} = 3.273\Omega$, $v = 9.091\text{V}$)

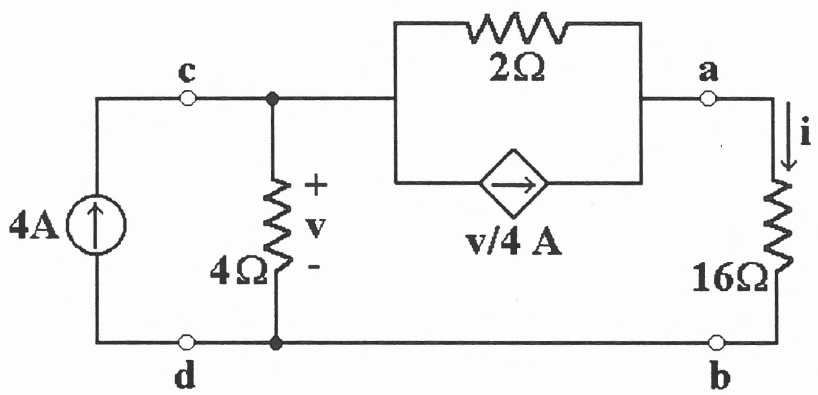


Figure 4.6: Figure for Question 4.7

4.8

Find the power delivered to R in Figure 4.7 when:

(a) R= 4Ω

(b) R= 12Ω

(c) R receives the maximum possible power from the circuit

(16W, 17.3W, 18W)

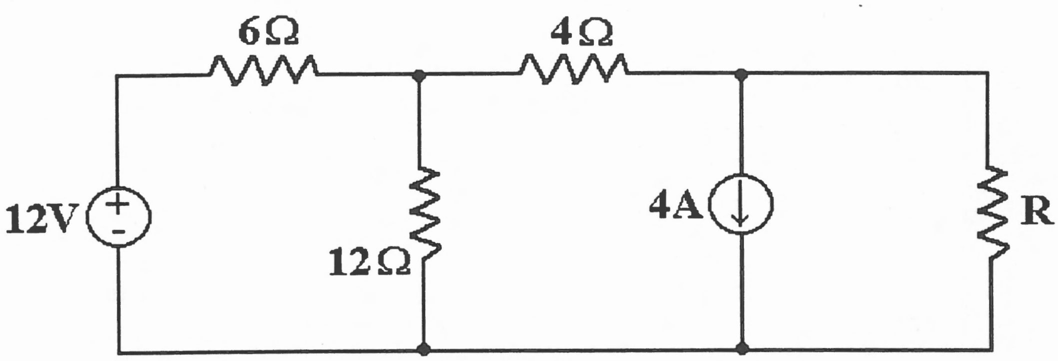


Figure 4.7: Figure for Question 4.8

4.9
Find the maximum power delivered to R in Figure 4.8 if:

(a) $R_1 = 30\Omega$
(b) $R_1 = 12\Omega$
(1.75W, 0W)

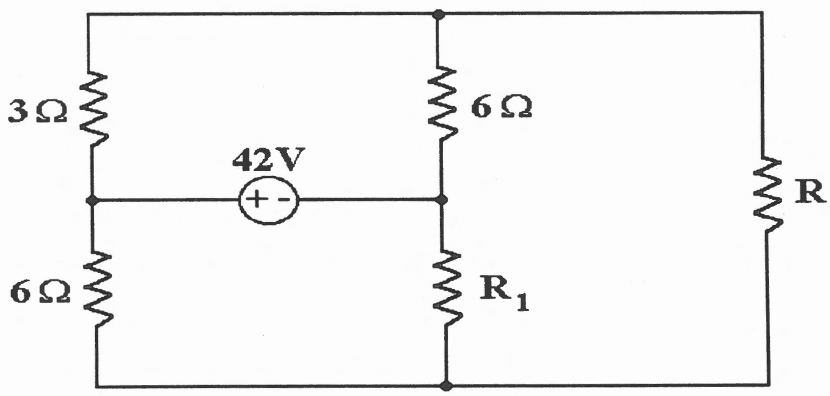


Figure 4.8: Figure for Question 4.9

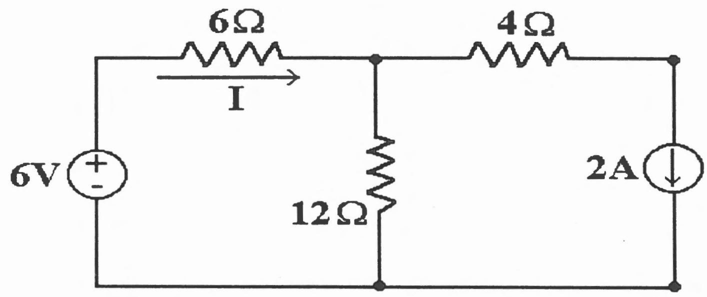
4.10
Prove that when a real current source i_g with internal shunt resistance R_p drives a load R_L, the maximum power transfer into the load occurs when $R_L = R_p$. What is the value of the maximum power transferred?
$(P_{MAX} = i_g^2 R_p/4)$

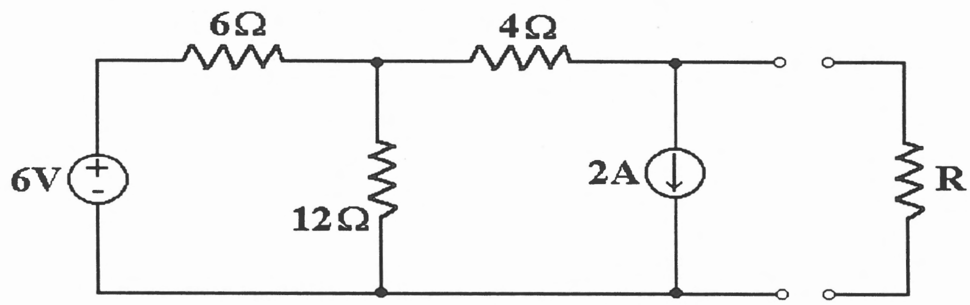
WORKSHEET 4

4.A

(a) Briefly say what is meant by the *Principle of Proportionality.*

(b) Use the *Principle of Superposition* to find the current I that flows in the 6Ω resistor in the circuit immediately below.





(c) If a load resistance R is now added to the circuit as shown immediately above, find the *Thévenin equivalent resistance* seen by R looking into the circuit.

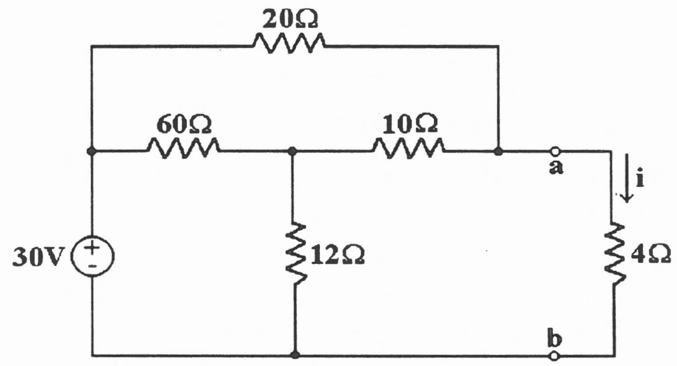
(d) What value must R be to receive maximum power from the circuit?

4.B

(a) Redraw the circuit below, with the components to the left of terminals a and b replaced with their Norton equivalent.

(b) Hence, find the power dissipated in the 4Ω resistor.

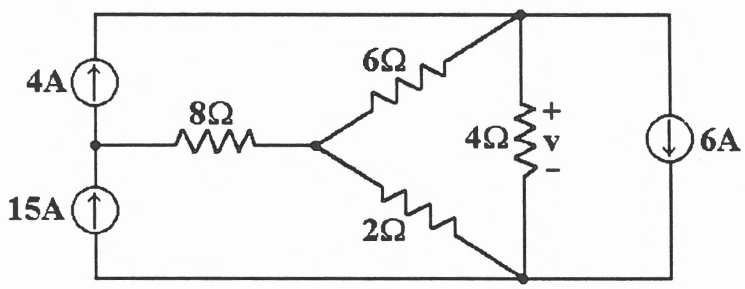
(c) State what value of load resistance connected across terminals a and b would absorb the greatest possible amount of power from the circuit to the left of the terminals.



4.C

Find v in the circuit below by replacing all of the circuit except the 4Ω resistor by its Thévenin equivalent.

In finding v_{oc}, you are advised to consider the **currents** *that flow in the circuit, rather than attempting nodal or mesh analysis*



4.D*

(a) Explain clearly and briefly what is meant by:

(i) the *Principle of Proportionality.*

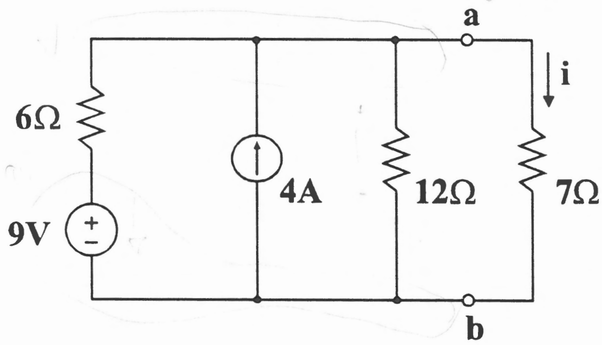
(ii) the *Principle of Superposition.*

(b) With reference to the figure below:

(i) replace everything in the network that is to the left of the terminals a and b by its Thévenin equivalent circuit, and then use the result to find the current i; $V_{oc} = 22V$, $R_{th} = 6\Omega$, $i = 2A$

(ii) confirm your answer to part (i) using the Principle of Superposition, and being careful to show all of your working.

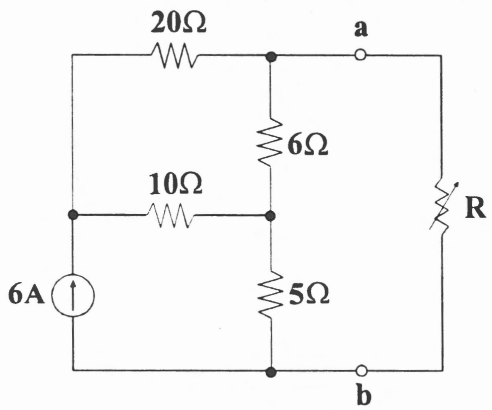
$V = 22V$



4.E*

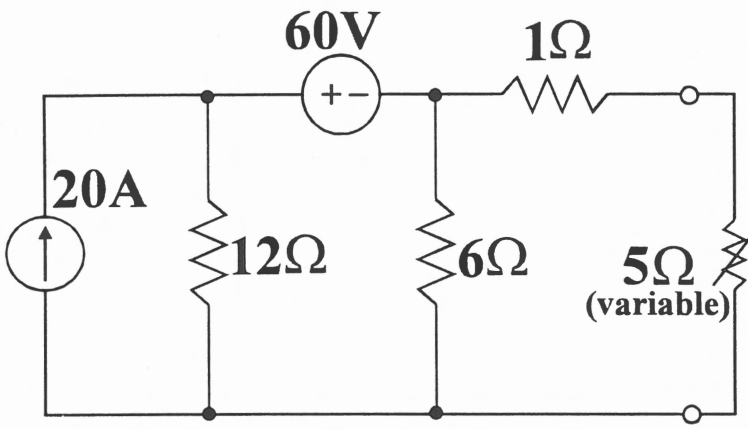
(a) Re-draw the circuit below, with the components to the left of terminals a and b replaced by their Norton equivalent. $R_{th} = 10\Omega$ $i_{sc} = 3A$ $6A$

(b) How could you arrange for the rest of the circuit to transfer the maximum possible power to the variable resistor R, and what would the value of that maximum possible power be? $P_{max} = \frac{i^2 R_{th}}{4} = 225W$



4.F

(a) Use *either* the Principle of Superposition *or* Thévenin's Theorem to find the current i that flows in the 5Ω load in the circuit below.



(b) Given that the load is a variable resistor presently set at 5Ω, state and explain clearly how the power delivered to the 5Ω load would change if the load resistance were either increased or decreased.

Chapter 5

Energy Storage Elements

In Lectures C9 and C10 we cover:

- The structure and behaviour of capacitors and inductors
- Formulæ for *energy storage* in capacitors and inductors
- Connecting capacitors or inductors in series or parallel
- An important condition in a circuit known as the *dc steady state*

5.1 Capacitors

If you make a break in a circuit and spread out the two terminals into large conducting plates, separating the plates with an insulator called a *dielectric*, clearly ordinary current can no longer flow through that part of the circuit. Instead, if you put a voltage across this new device, charge will tend to pile up on the plates (positive charge on one plate and negative charge on the other).

We could imagine a small charge $+\Delta q$ transferred onto the top plate in Figure 5.1 with $-\Delta q$ transferred to the bottom plate. The work done in moving these charges is Δv. The more charge that is transferred, the higher the voltage will be across the plates, and this works both ways, so that if you apply a voltage across the plates, the corresponding transfer of charge will occur. The voltage and the transferred charge are always proportional for any particular pair of plates, so if the plates are charged to $\pm q$ coulombs

and the voltage across them is v volts, then we can write $q = Cv$, where C, the constant of proportionality, is the *capacitance* of the device, which we call a *capacitor*. The SI unit of capacitance is the *farad*, and $1F = 1C/V$. You will immediately notice that capacitors are *linear* devices. The way

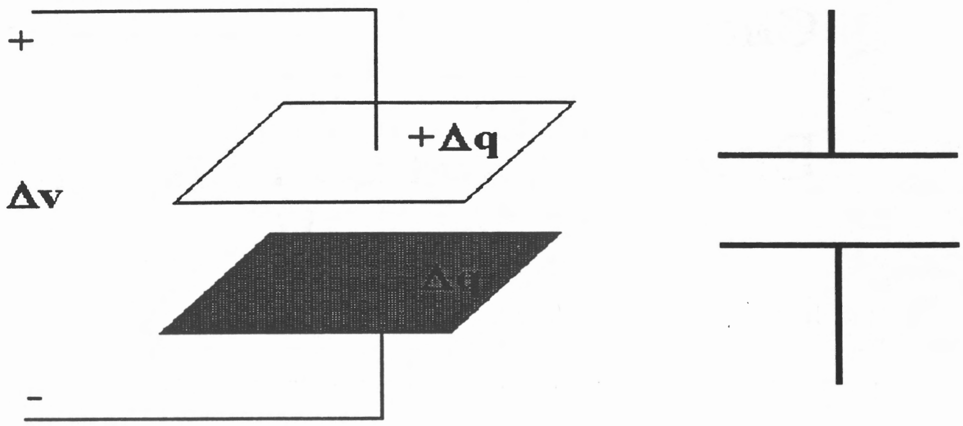


Figure 5.1: A Simple Parallel Plate Capacitor and its Symbol

that they relate current to voltage can be seen by differentiating $q = Cv$ to get

$$i = C\frac{dv}{dt}$$

From this equation we note:

- The faster the voltage changes, the greater the current that flows through the capacitor. A capacitor only acts as an open circuit to a dc voltage input (for which $\frac{dv}{dt} = 0$). Thus *capacitors block dc* but will pass an ac current in response to an ac voltage across their inputs.

- Since infinite current cannot flow, it is not possible for the capacitor voltage to change in zero time (i.e. instantaneously). Current through a capacitor may be discontinuous, but the voltage across it is *continuous.*

- We can re-write the capacitor equation as $v(t) = 1/C \int_{t_i}^{t_f} idt$, so the voltage at time t is $v(t) = 1/C \int_{-\infty}^{t} idt = 1/C \int_{t_o}^{t} idt + v(t_o)$, where $v(t_o) = q(t_o)/C$ is the voltage on the capacitor at time t_o. Note that $v(-\infty) = 0$.

$$v(t) = \frac{1}{C}\int_{t_i}^{t_f} i \cdot dt$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i \cdot dt = \frac{1}{C}\int_{t_o}^{t} i \cdot dt + \underline{v(t_o)}$$

$$v(t_o) = q(t_o)/C$$

5.2 Energy Storage in Capacitors

Recalling that $1V = 1J/C$, we say that work is done by the applied voltage in separating the charges on the capacitor plates. This energy is not dissipated (as in a resistor) but goes to form an *electric field* between the plates. The energy is stored in the electric field, and can be recovered from it. The amount of energy stored is $w_c(t) = \int_{-\infty}^{t} vidt = \int_{-\infty}^{t} v\left(C\frac{dv}{dt}\right)dt = C\int_{-\infty}^{t} vdv = \frac{1}{2}Cv^2(t)|_{t=-\infty}^{t}$, and since $v(-\infty) = 0$

$$w_c(t) = \frac{1}{2}Cv^2(t) \text{ in Joules.}$$

Using $v = q/C$, this can be written $w_c(t) = \frac{1}{2}q^2(t)/C$.

Note that $w_c(t)$ is always positive, so capacitors are *passive* elements, and that since $v(t)$ is necessarily continuous, $w_c(t)$ is also continuous (the energy stored in a capacitor cannot jump up or down instantaneously).

5.3 Series and Parallel Capacitors

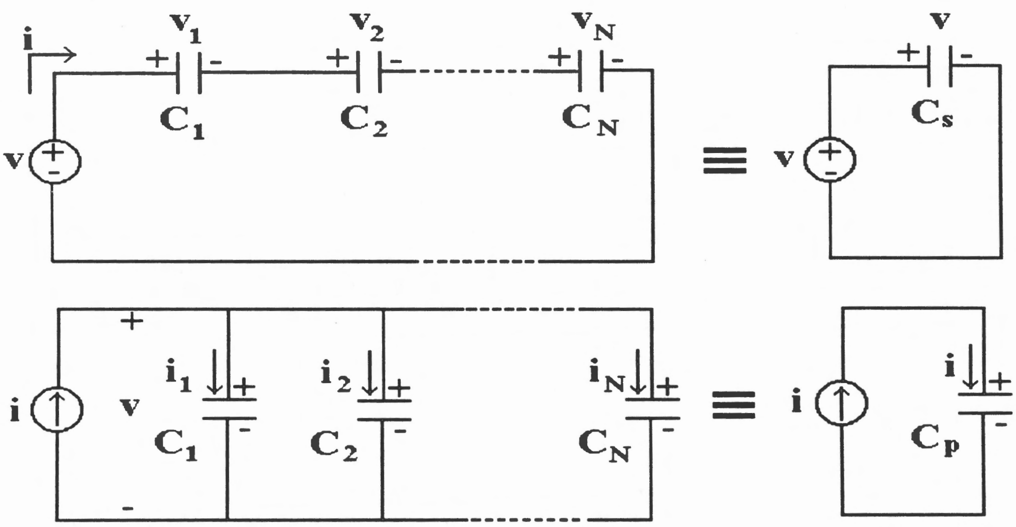


Figure 5.2: Equivalents for Capacitors in Series and in Parallel

If we connect N capacitors in series with a voltage source (as in Figure 5.2), then a current will flow, and a voltage will appear across each capacitor, depending on its capacitance, with $v_k(t) = 1/C_k \int_{t_o}^{t} idt + v_k(t_o)$.

Then, by KVL:

$$
\begin{aligned}
v &= v_1 + v_2 + \ldots + v_N \\
&= 1/C_1 \int_{t_o}^{t} idt + v_1(t_o) + 1/C_2 \int_{t_o}^{t} idt + v_2(t_o) + \ldots + 1/C_N \int_{t_o}^{t} idt + v_N(t_o) \\
&= (1/C_1 + 1/C_2 + \ldots + 1/C_N) \int_{t_o}^{t} idt + v_1(t_o) + v_2(t_o) + \ldots + v_N(t_o) \\
&= \left(\sum_{n=1}^{N} 1/C_n \right) \int_{t_o}^{t} idt + v(t_o)
\end{aligned}
$$

Compare to this the voltage across a single *equivalent series capacitor*, C_s, which is $v = 1/C_s \int_{t_o}^{t} idt + vt_o$, and it is now seen that

$$1/C_s = 1/C_1 + 1/C_2 + \ldots + 1/C_N$$

This is just like the rule for parallel resistors, and a "product over sum" rule applies for two capacitors in series, as you might expect. If, instead, we connect N capacitors in parallel (see Figure 5.2), and then drive them with a current source, a separate current will flow through each capacitor, given by $i_k = C_k \frac{dv}{dt}$. Using KCL, we then obtain:

$$
\begin{aligned}
i &= i_1 + i_2 + \ldots + i_N \\
&= C_1 \tfrac{dv}{dt} + C_2 \tfrac{dv}{dt} + \ldots + C_N \tfrac{dv}{dt} \\
&= (C_1 + C_2 + \ldots + C_N) \tfrac{dv}{dt} \\
&= \sum_{n=1}^{N} C_n \frac{dv}{dt}
\end{aligned}
$$

Compare this to the current through an *equivalent parallel capacitor*, C_p, which is $i = C_P \frac{dv}{dt}$, and it is clear that

$$C_p = C_1 + C_2 + \ldots + C_N$$

You will see that this is similar to the rule for resistors in series.

5.4 Inductors

Just as static charges give rise to an electric field, in which capacitors store their energy, moving charges (i.e. currents) give rise to a *magnetic field.* Magnetic fields can be detected near a current-carrying wire using a small compass, and they can also store energy, as we shall see.

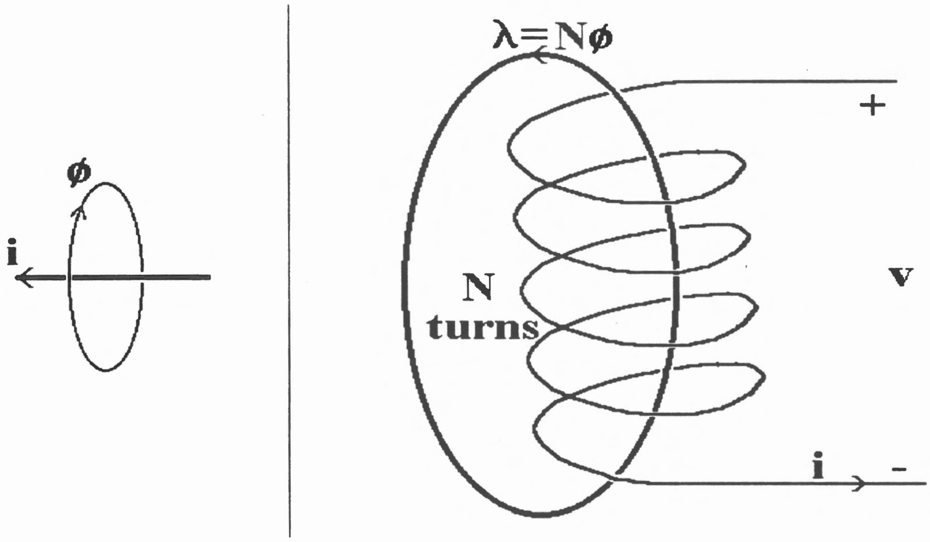


Figure 5.3: Magnetic Flux, and Flux Linkage in a Coil of N Turns

A single wire that carries current produces a *magnetic flux*, ϕ, measured in Webers (Wb), which can tell us the strength of the magnetic field around the wire. If we want a stronger field, we could call for more current, or have more wires, each carrying the same current. We could also *coil* the current-carrying wire, because each individual coil would then contribute magnetic flux. which would add up to give a total *magnetic flux linkage* of $\lambda = N\phi$, if there were N coils.

The higher the current through the coil the greater the total flux linkage, and we can express this as a statement of proportionality: $\lambda = Li$, where L, the constant of proportionality, is termed the *inductance* of the coil, which is itself known as an *inductor.* The unit of inductance is the *henry*, and $1H = 1Wb/A$. Inductors, like capacitors and resistors, are *linear* devices.

Now, *Faraday's Law of Electromagnetic Induction* states that if the magnetic

flux (λ) ever changes, then a voltage will be *induced.* This could happen if you moved the current-carrying coil around near a piece of wire (a voltage would be induced in the wire causing current to flow), or indeed if you varied the current going throught the coil (because that would also produce a change in magnetic field). Either way, the voltage induced is related to the flux linkage by $v = \frac{d\lambda}{dt}$. Putting this together with $\lambda = Li$, we get

$$v = L\frac{di}{dt}$$

From this equation we note:

- If the current through the inductor is dc, the voltage across it will be zero, but if the current changes rapidly (e.g. high frequency ac), then the voltage across the inductor will be high. *Inductors act like a short circuit to dc*

- Any instantaneous change in current would require an infinite voltage to exist across the device, which is impossible. Therefore, the current through an inductor must be continuous.

- We can re-write the inductor equation as $i(t) = 1/L \int_{t_i}^{t_f} v(t)dt$, so the current at time t is $i(t) = 1/L \int_{-\infty}^{t} v(t)dt = 1/L \int_{t_o}^{t} v(t)dt + i(t_o)$, where $i(t_o)$ is the inductor current at time t_o. Note that $i(-\infty) = 0$.

It is important to realise that as the current through an inductor increases, a voltage develops across its terminals which tends to *oppose* the build-up of current (not to aid it). This is *Lenz's Law*, and it makes the polarity markings on inductors in circuits unambiguous (see Figure 5.3).

5.5 Energy Storage in Inductors

Work is performed to establish the flux linkage in an inductor, and this becomes the energy which is stored in the inductor's magnetic field. We can calculate this stored energy as follows: $w_L(t) = \int_{-\infty}^{t} vidt = \int_{-\infty}^{t}(L\frac{di}{dt})idt = L\int_{-\infty}^{t} idi = \frac{1}{2}Li^2(t)|_{t=-\infty}^{t}$, and since $i(-\infty) = 0$,

$$w_L(t) = \frac{1}{2}Li^2(t) \text{ in Joules.}$$

Once again, this energy is always positive, so the inductor is a *passive* element. No energy is lost in the device, so there is no dissipation of power (unlike in a resistor). Note also that since $i(t)$ is continuous, $w_L(t)$ cannot change instantaneously, but is also continuous.

5.6 Series and Parallel Inductors

If we connect N inductors in series with a voltage source (see Figure 5.4), then a current will flow, and a voltage will appear across each inductor, depending on its inductance, with $v_k(t) = L_k \frac{di}{dt}$.

Then, by KVL:

$$\begin{aligned} v &= v_1 + v_2 + \ldots + v_N \\ &= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \ldots + L_N \frac{di}{dt} \\ &= (L_1 + L_2 + \ldots + L_N) \frac{di}{dt} \\ &= \left(\sum_{n=1}^{N} L_n \right) \frac{di}{dt} \end{aligned}$$

Compare to this the voltage across a single *equivalent series inductor*, L_s, which is $v = L_s \frac{di}{dt}$, and it is now seen that

$$L_s = L_1 + L_2 + \ldots + L_N$$

This is just like the rule for series resistors.

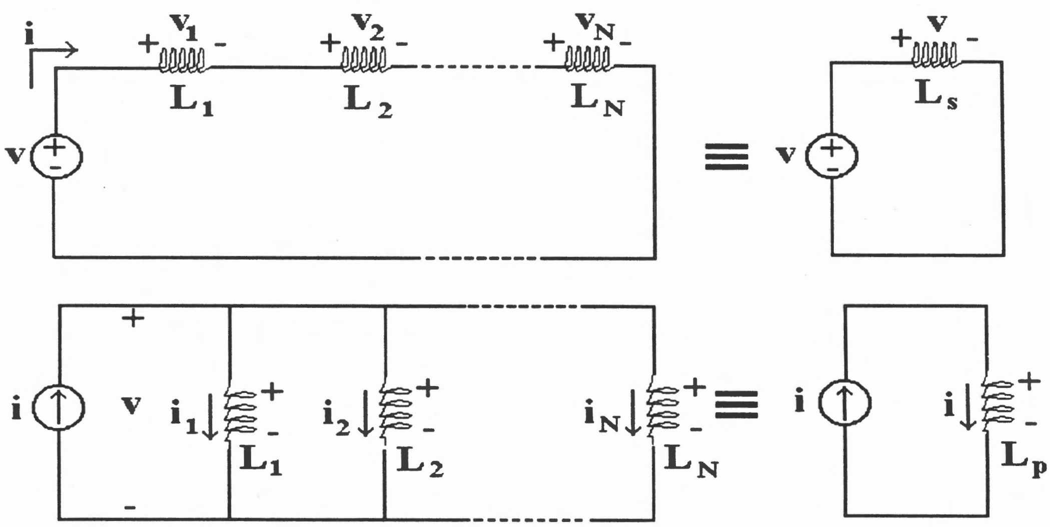


Figure 5.4: Equivalents for Inductors in Series and in Parallel

If, instead, we connect N inductors in parallel (refer again to Figure 5.4), and then drive them with a current source, then a separate current will flow through each inductor, given by $i_k = 1/L_k \int_{t_o}^{t} v dt + i_k(t_o)$.

Using KCL, we then obtain:

$$\begin{aligned}
i &= i_1 + i_2 + \ldots + i_N \\
&= 1/L_1 \int_{t_o}^{t} v dt + i_1(t_o) + 1/L_2 \int_{t_o}^{t} v dt + i_2(t_o) + \ldots + 1/L_N \int_{t_o}^{t} v dt + i_N(t_o) \\
&= (1/L_1 + 1/L_2 + \ldots + 1/L_N) \int_{t_o}^{t} i dt + i_1(t_o) + i_2(t_o) + \ldots + i_N(t_o) \\
&= \left(\sum_{n=1}^{N} 1/L_n \right) \int_{t_o}^{t} v dt + i(t_o)
\end{aligned}$$

Compare this to the current through an *equivalent parallel inductor*, L_p, which is $i = 1/L_P \int_{t_o}^{t} v dt + i(t_o)$, and it is clear that

$$1/L_p = 1/L_1 + 1/L_2 + \ldots + 1/L_N$$

You will see that this is similar to the rule for resistors in parallel, and in the special case of two inductors in parallel, the "product over sum" rule applies in finding their equivalent inductance.

5.7 The DC Steady State

Very frequently, we shall be interested in analysing the behaviour of circuits which contain capacitors and/or inductors, and this often means predicting how currents and voltages in the circuit will respond to a specified change in the network (e.g. the closing of a switch which brings another source into the circuit). Quite clearly, there is not much hope of analysing changes in the currents and voltages unless we know what those quantities were doing immediately before the switch was thrown (or whatever change was made).

If all the sources in the circuit are dc voltages or currents, then *over a long time* all of the currents and voltages within the circuit will settle down to steady values. You can see that this must be the case, because the principle of superposition holds, since all of the circuit elements (resistors, capacitors

and inductors) are *linear*, and hence the circuit is linear too. If all the sources are dc, the sum of their contributions to a given voltage or current must be dc. However, since capacitors and inductors store energy, it may take some time for swapping of energy between energy storage elements to cease. We shall see later what "some time" really means.

For now, it should be clear that if we throw a switch in a circuit, then provided the circuit has been unchanged for some time before, all its voltages and currents will be in a *steady state* at $t = 0$ when the switch is thrown. Since capacitors are open circuits to dc, and inductors are short circuits, it is usually very easy to analyse the steady state, because all we are left with is a network of resistors and dc sources. This enables us to work out what conditions were like *just before* the switch was thrown (at $t = 0^-$). Once we have that information, we have the *initial conditions* for our real problem, which is to predict the behaviour of the circuit from $t = 0^+$.

TUTORIAL 5

5.1

(a) A 100nF capacitor has a voltage across its terminals of $6\cos(500t)$ V. Find the current in the capacitor as a function of time.

(b) A constant current of 20mA is discharging a 22μF capacitor (leaving its positive voltage terminal). If the capacitor was initially charged to 6V, find the charge and voltage on it after 20ms.

(c) A 100nF capacitor has a charge of 20μC. Find the size of the voltage across the capacitor, and the energy stored in it.

(d) If the energy stored in a 100mF capacitor is 10J, find the size of the voltage across the capacitor, and the charge on the capacitor.
($-300\sin(500t)\mu$A; -268μC, -12.2V; 200V, 2mJ; 14.1V, 1.41C)

5.2

(a) In the circuit of Figure 5.5, the switch has been open and is closed at $t = 0$. If the current in R_2 at $t = 0^-$ is 2A directed downwards, list the values of the following quantities at $t = 0^-$ and at $t = 0^+$:

(i) the charge on the capacitor
(ii) the current in R_1, directed to the right
(iii) the current in C, directed downwards
(iv) the value of dv_c/dt

(2C, 2C; 0A, 3A; -2A, 1A; -8V/s, 4V/s)

(b) Repeat part (a) if the switch has been closed for a long time and is then opened at $t = 0$. The current in R_2 has reached 2.5A downwards at $t = 0^-$.
(2.5C, 2.5C; 2.5A, 0A; 0A, -2.5A; 0V/s, -10V/s)

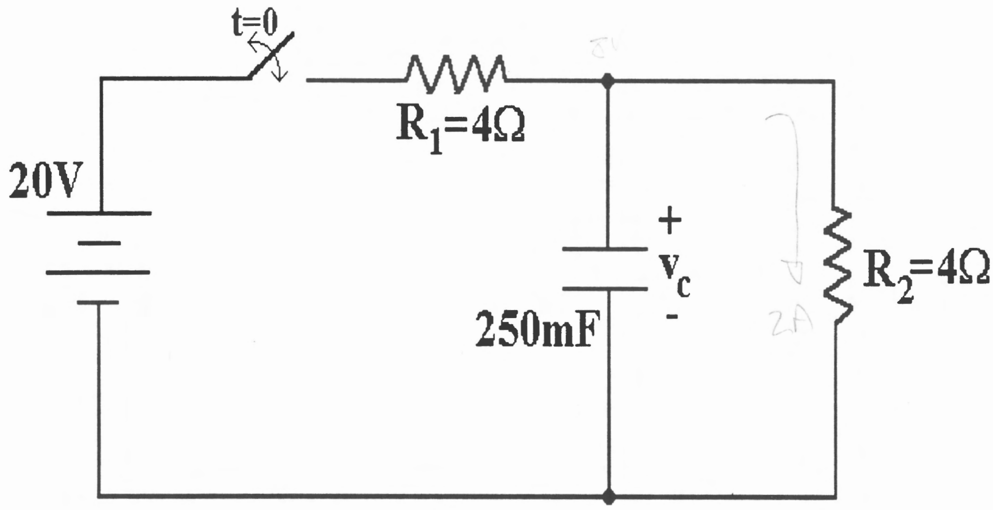


Figure 5.5: Figure for Question 5.2

5.3

(a) What are the maximum and minimum values of capacitance that can be obtained from twelve 1μF capacitors?

(b) Derive an equation for current division between two parallel capacitors by finding i_1 and i_2 in Figure 5.6(a).

(c) Derive an equation for current division between two parallel inductors by finding i_1 and i_2 in Figure 5.6(b).
(12μF, 83.3nF; $i_1 = \frac{C_1}{C_1+C_2}i$, $i_2 = \frac{C_2}{C_1+C_2}i$; $i_1 = \frac{L_2}{L_1+L_2}i$, $i_2 = \frac{L_1}{L_1+L_2}i$)

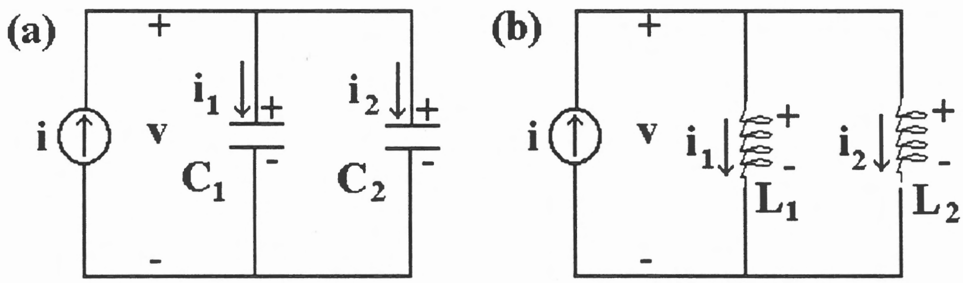


Figure 5.6: Figures for Question 5.3

5.4

(a) A 10mH inductor has a current of $50\cos(1000t)$ mA. Find its voltage and its flux linkage.
$(-500\sin(1000t)$ mV, $500\cos(1000t)$ μWb)

(b) Find the current $i(t)$ for $t > 0$ in a 20mH inductor having a voltage of $4\sin(10t)$ V, given that $i(0) = -20$ A.
$(-20\cos(10t)$ A)

5.5

(a) Derive a simple expression for the energy stored in an inductor in terms of the flux linkage λ and the inductance L.

(b) A 20mH inductor has a current $i = 5\cos(50\pi t)$ mA. Find the flux linkage and the energy that is stored, both at $t = 40$ms.

(c) A 50μH inductor has a voltage $v = 3\sin(2000t)$ V, with $i(0) = 2.5$ A. Find the energy stored in the inductor at $t = \pi/6$ ms.
($w_L = \lambda^2/2L$; 100μWb, 250nJ; 7.66mJ)

5.6

In Figure 5.7, let I = 5A, R_1 = 6Ω, R_2 = 4Ω, L = 2H, and $i_1(0^-)$ = 2A. If the switch is open until $t = 0^-$ and closed at $t = 0$, find $i_L(0^-)$, $i_L(0^+)$, $i_1(0^+)$ and $di_L(0^+)/dt$.
(3A, 3A, 0A, -6A/s)

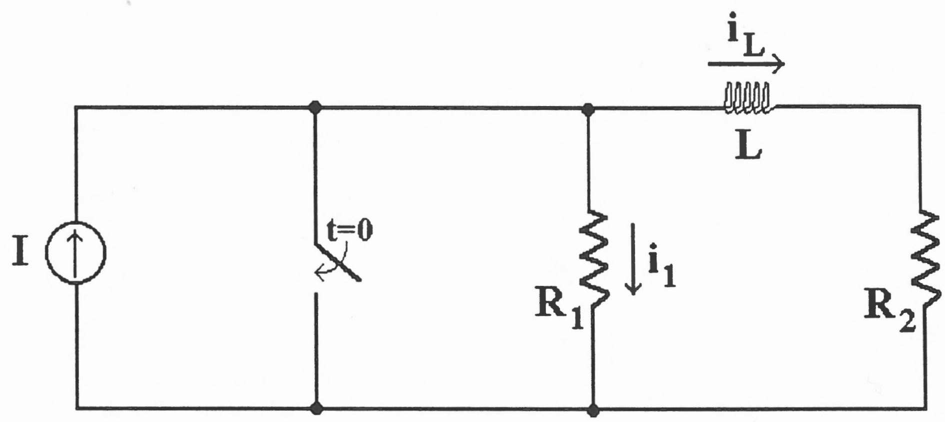


Figure 5.7: Figure for Question 5.6

5.7
The circuit of Figure 5.8 is in dc steady state at $t = 0^-$.
Find, at both $t = 0^-$ and $t = 0^+$: (a) i_1, (b) i_2, (c) i_3, (d) i_c, (e) v_c.
(2A, 8A; 2A, -4A; 2A, 2A; 0A, -6A; 12V, 12V)

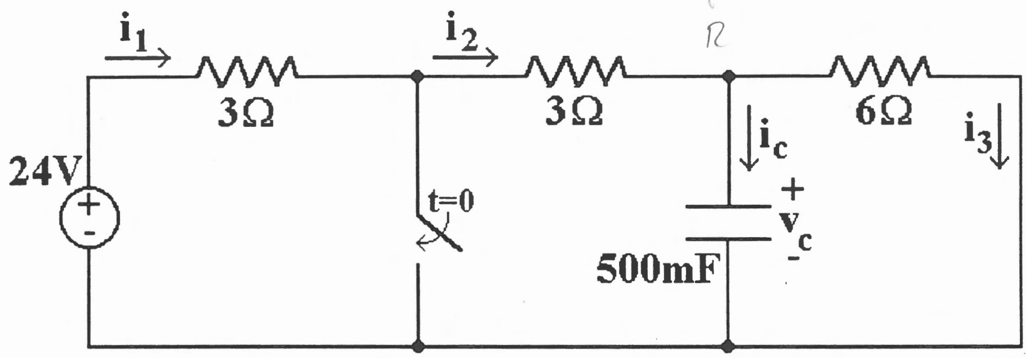


Figure 5.8: Figure for Question 5.7

5.8
If the circuit of Figure 5.9 is in dc steady state at $t = 0^-$, find the following at both $t = 0^-$ and $t = 0^+$: (a) i_1, (b) i_L, (c) v_L
(4A. -2A; 2A, 2A; 0V, -36V)

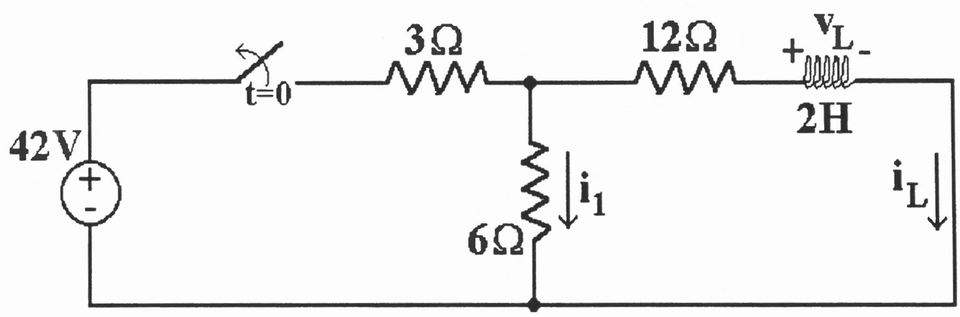


Figure 5.9: Figure for Question 5.8

5.9

The circuit of Figure 5.10 is in dc steady state at $t = 0^-$.
Find: (a) v_c, (b) i_L, (c) i, (d) i_R at both $t = 0^-$ and $t = 0^+$.
(8V, 8V; 4A, 4A; 4A, 4A; 1A, -4A)

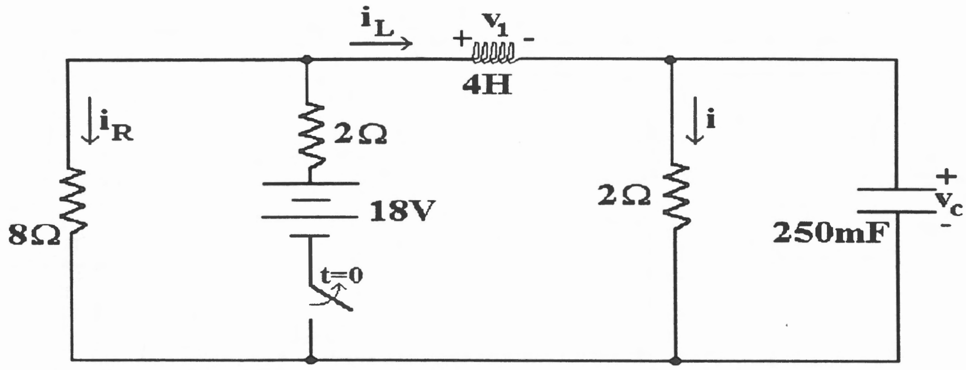


Figure 5.10: Figure for Question 5.9

5.10

If the circuit of Figure 5.11 is in dc steady state at $t = 0^-$, find v_1 and v_2 at both $t = 0^-$ and $t = 0^+$.
(10V, 50V; -40V, 40V)

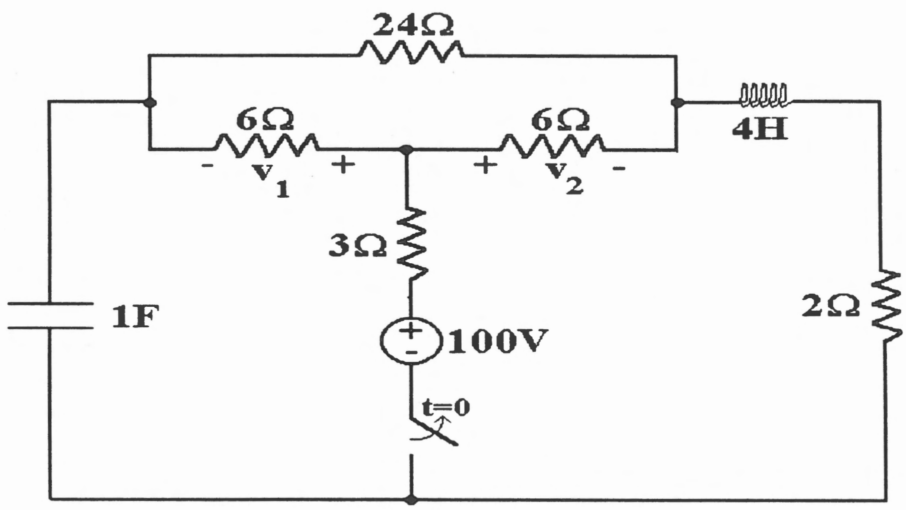
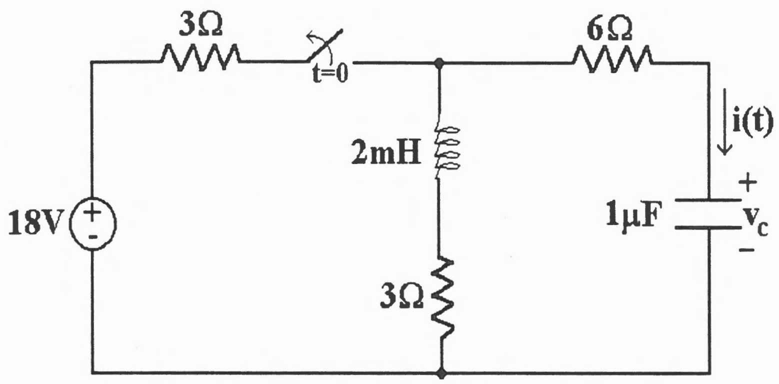


Figure 5.11: Figure for Question 5.10

WORKSHEET 5

5.A

In the circuit shown, the switch has been closed for a long time, so the circuit is in the dc steady state at $t = 0^-$.



(a) In this state, find:

(i) The voltage across the capacitor

(ii) The charge on the capacitor

(iii) The energy stored in the capacitor

(iv) The energy stored in the inductor

(v) The voltage across the inductor

(vi) The flux linkage in the inductor

(b) The switch is opened at time $t = 0$.

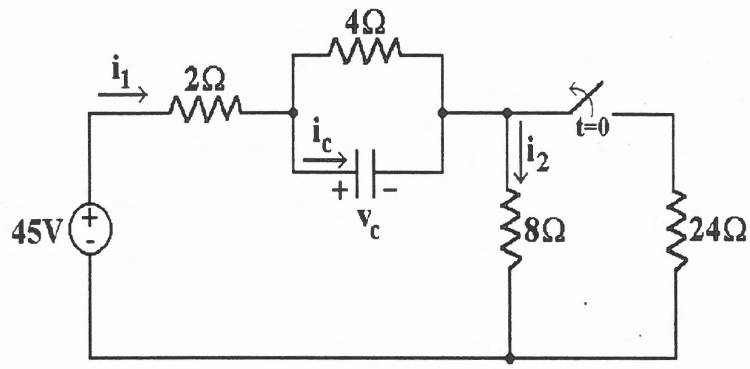
(i) Find $i(0^+)$

(ii) Find $\dfrac{di(0^+)}{dt}$

(iii) Find (in standard form), but **do not attempt to solve**, the second-order differential equation from which $i(t)$ could be calculated for $t > 0$.

5.B

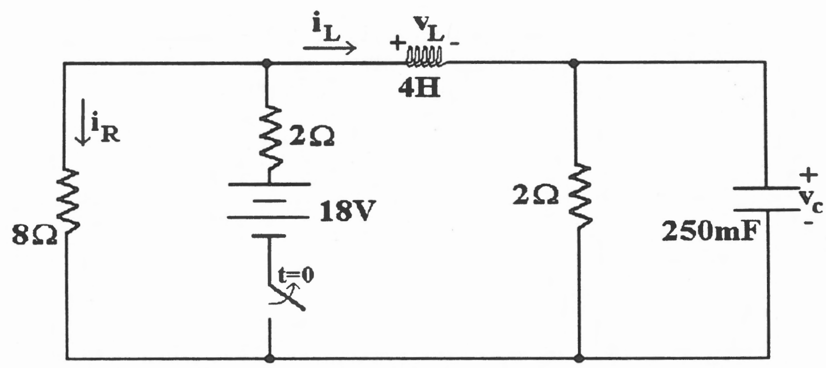
In the circuit given below, if $i_1(0^-) = \frac{15}{4}$A, then find (in any order) i_1, i_2, i_c and v_c at $t = 0^+$. Note that the given initial condition is **not** the dc steady state.



5.C

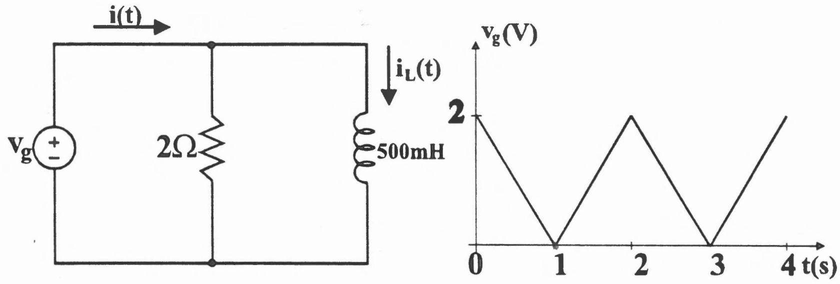
The switch in the circuit shown below has been closed for a long time, so that the circuit is in the dc steady state at $t = 0^-$, just before the switch is opened. Find, at both $t = 0^-$ and at $t = 0^+$:

(a) The charge on the capacitor, q_c
(b) The current in the 8Ω resistor, i_R
(c) The voltage across the inductor, v_L
(d) The flux linkage in the inductor, ϕ_L
(e) The power supplied by the 18V battery, P_s



5.D*

The figure below shows a circuit and also gives the source voltage, v_g, as a function of time. Given that $i_L(0) = -1$A, find *and sketch* the current $i(t)$ for $0 < t < 4$s.



5.E*

The switch in the circuit shown below is opened at time $t = 0$. Given that an instant before the switch opens the voltage across the $3\,\Omega$ input resistor is $v(0^-) = 9\,$V, and also that $i_c(0^-) = 1\,$A, find at $t = 0^+$:

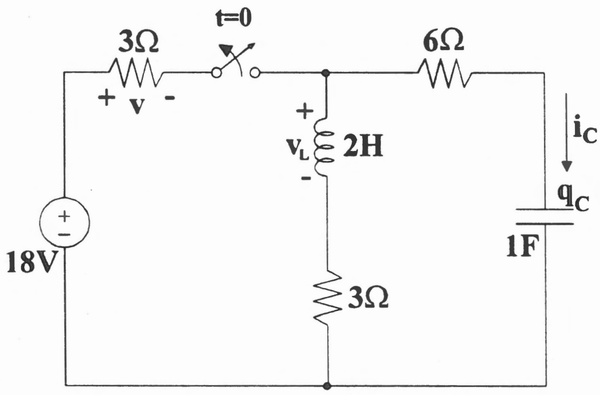
(a) The charge on the capacitor, $q_c(0^+)$;

(b) The current in the capacitor, $i_c(0^+)$;

(c) The voltage across the inductor, $v_L(0^+)$;

(d) The flux linkage in the inductor, $\lambda_L(0^+)$;

(e) The value of $\dfrac{di_c(0^+)}{dt}$.



5.F

NO WORKING IS REQUIRED IN ANSWERING THIS QUESTION.
A VERY SHORT ANSWER WILL BE SUFFICIENT IN EACH CASE.
EACH CORRECT ANSWER (WITH UNITS) CARRIES 2 MARKS.

(a) What is the value of a resistor whose colour code is
GREY RED YELLOW BROWN

(b) How would you express the value of a 6910000pF capacitor in correct engineering notation?

(c) A small blue resin-dipped monolithic ceramic capacitor is marked "104". What is its capacitance?

(d) State the time constant of a source-free LR circuit, in which $L = 20$mH and $R = 800\Omega$.

(e) A decaying exponential falls from an initial value of 100V. How long does it take to reach 2V if its time constant is 5s?

(f) What is the total flux linkage due to a 2A current flowing in a 4H inductor?

(g) A circuit consists of 4 elements connected in parallel - a 3F capacitor, a 1H inductor, a 5H inductor and a 6F capacitor. Reduce these four elements to two.

(h) State the resonant frequency of an RLC circuit in which $R = 13\Omega$, $L = 4$mH and $C = 25$nF.

(i) What is the form of the solution to a homogeneous equation which represents the natural response in the critically damped case?

(j) If the forcing function in a circuit is $2\cos(3t)$, what should the trial solution be in the calculation of forced response?

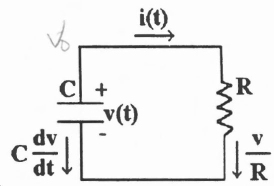
Chapter 6

RC and RL Circuit Responses

In Lectures C11 and C12 we cover:

- Finding the *response* over time in *source-free RC circuits*
- The concept of the *time constant*
- Voltages and currents in *source-free RL circuits*
- The response to a *constant forcing function* in RC or RL circuits
- The response to a general forcing function in circuits containing one energy storage element

6.1 Source-free RC circuits



Consider this small RC circuit. If the capacitor is charged to V_o at $t = 0$, then the circuit response after $t = 0$ depends *entirely* on the energy stored in the capacitor, which is $w_c(0) = \frac{1}{2}CV_o^2$.

Now, by KCL, $C\frac{dv}{dt} + v/R = 0$, so $\frac{dv}{dt} + (1/RC)v = 0$. This is a *first order differential equation* of a simple sort, which can be solved by *separating the variables*, like this:

$$\frac{dv}{dt} = -\frac{1}{RC}v \qquad \text{so} \qquad \frac{dv}{v} = -\frac{1}{RC}dt \qquad \text{so} \qquad \int \frac{dv}{v} = -\frac{1}{RC}\int dt$$

Standard form: $\frac{dv}{dt} + \frac{v}{RC} = 0$ Must get it into this form

handkerchief on computer screen

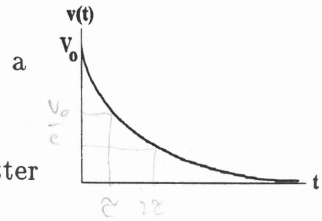
Two radioactive isotopes
1 with half life of 33 minutes
another 26 minutes

Integration now gives $\ln|v| = -\frac{t}{RC} + K$, where K is a *constant of integration.* to be found

At $t = 0$, $v = V_o$, and so we can substitute into the last equation to get $\ln|V_o| = K$, and then re-write our equation for v as:

$$\ln|v| = -\frac{t}{RC} + \ln|V_o| \qquad \text{or} \qquad \ln\left(\frac{v}{V_o}\right) = -\frac{t}{RC} \qquad \text{or} \qquad \frac{v}{V_o} = e^{-t/(RC)}$$

Hence, finally, $v(t) = V_o e^{-t/(RC)}$. In other words, we now know the voltage across the capacitor as a function of time. Note that the current in the resistor is now easy to obtain. It is $i(t) = \frac{V_o}{R} e^{-t/(RC)}$.



The voltage function we have calculated looks like this. This is a familiar function, known as an *exponential decay*, in which the voltage will tend to zero over a long period of time, but will be reduced by a constant ratio in any fixed period of time, no matter when this fixed period occurs.

Notice that this response of the circuit depends *only* on the interaction of its passive elements, and is not caused by any independent current or voltage source. It is therefore known as the *natural response* of the circuit. Note also that, in general, if the capacitor begins to discharge through the resistor at $t = t_o$, then the voltage response is given by $v(t) = V_o e^{-(t-t_o)/(RC)}$ for $t > t_o$.

6.2 Time Constants

To describe *how rapidly* a decaying exponential falls towards zero (or any exponential moves towards its limiting value), we use the concept of the *time constant.* For radioactive decay, which is also exponential, you will probably have heard the term "half-life", which is the period of time required for the radioactivity to fall to 1/2 of its initial value. In electrical engineering, we find it more natural to speak of the "time constant" (τ) of the decay curve, which is *defined* as the time required for the voltage or current to fall to $1/e$ of its initial value. Here $e = 2.71828182846...$ is the base of the natural logarithm.

In the case of the source-free RC circuit, the time taken for the capacitor voltage to fall from V_o to V_o/e is τ, where $V_o e^{-\tau/RC} = V_o e^{-1}$. We therefore see that $-\tau/RC = -1$ and so

$$\tau = RC$$

The following points about the time constant are of interest:

- The units of τ are $\Omega F = (V/A)(C/V) = (C/A) = s$.
- We can write $v(t) = V_o e^{-t/\tau}$. This means that, if we know the time constant, then we can write down the general form of the decaying response. Then, if we know the initial voltage (perhaps from the dc steady state conditions immediately prior to actuating a switch at $t = 0$), we can write down the expression for v(t) for $t > 0$ relatively quickly.
- At the end of *two* time constants, the voltage and current have fallen to $1/e^2$ of their initial values, and they are only at V_o/e^3 and I_o/e^3 when $t = 3\tau$, and so on. As a percentage of initial value, this decay is tabulated below:

At time	Percent of Initial Value
τ	36.8
2τ	13.5
3τ	4.98
4τ	1.83
5τ	0.67

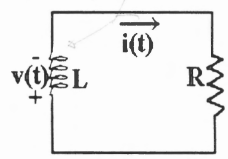
Tolerance of components ~~so~~ will be greater than this

Tolerance 'fallen through the noise floor'

- Engineers often regard an exponentially decaying quantity effectively to have vanished after *five* time constants

6.3 Source-free RL Circuits

arbitrary assignment of signs



Let us apply these techniques to the RL circuit shown.We suppose that at $t = 0$ the inductor has current I_o flowing in it, and so the energy stored in its magnetic field is $w_L(0) = \frac{1}{2}LI_o^2$.

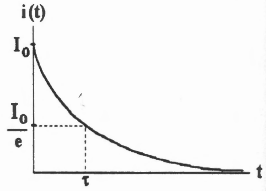
Using KVL, we obtain $-L\frac{di}{dt} - Ri = 0$, and so $\frac{di}{dt} + \frac{R}{L}i = 0$. Obviously, this could be solved by separation of the variables to get $i(t)$, but we present here an important alternative method of solution, which we will call for convenience the "s-method".

We know that $i(t)$ will be found to be an exponentially decaying function that begins at an initial value (call it A) and that dies away to zero over time. In other words, the final equation will look like $i(t) = Ae^{st}$, where A and s are as yet unknown. If we feed this proposed form for the solution back into the differential equation, we get

$$Ase^{st} + A\frac{R}{L}e^{st} = 0 \qquad \text{so} \qquad \left(s + \frac{R}{L}\right)Ae^{st} = 0$$

Solving for s, either $Ae^{st} = 0$, which is only possible if $A = 0$ (in which case the inductor had no energy in the first place, so the exercise is pretty pointless), or, if $s + \frac{R}{L} = 0$, then $s = -\frac{R}{L}$. In this case, we shall obtain $i(t) = Ae^{-(R/L)t}$, and , since $i(0) = I_o$, we have $i(t) = I_oe^{-(R/L)t}$ (which you could confirm by separation of the variables in the original equation).

If we recall that, by definition of the time constant, $i(t) = I_oe^{-t/\tau}$, we see that the time constant in the RL circuit is $\tau = L/R$. The units of τ this time are $H/\Omega = (Vs/A)/(V/A) = s$, as in the RC case. So the current function looks like this. Clearly, now, if we are interested in the voltage across R, all that we need to do is to apply Ohm's Law, to obtain $v(t) = I_oRe^{-t/\tau}$. It is interesting to note that, if we had tackled the circuit by applying KCL rather than KVL, then we would have obtained the *integral* equation:



$$-\frac{V}{R} - \frac{1}{L}\int_0^t vdt + i(0) = 0.$$

Differentiating this equation gives:

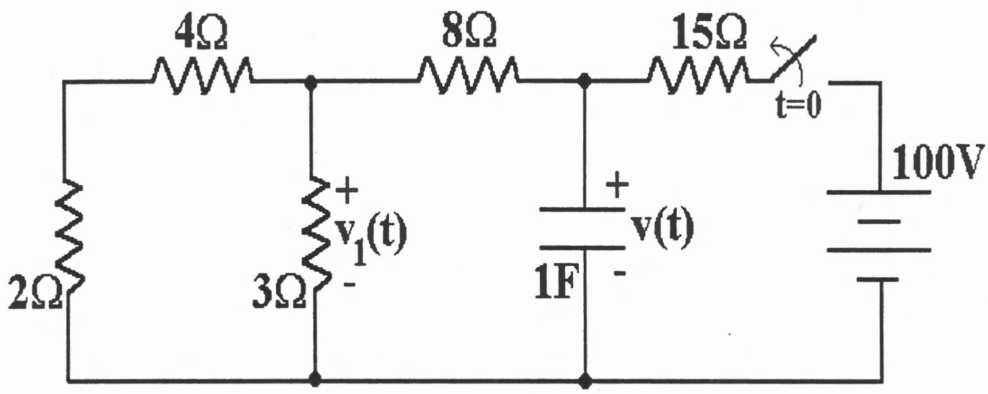
$$-\frac{1}{R}\frac{dv}{dt} - \frac{1}{L}v = 0 \qquad \text{so} \qquad \frac{dv}{dt} + \frac{R}{L}v = 0 \qquad \text{so} \qquad v(t) = V_oe^{-\frac{Rt}{L}},$$

which is the same as the voltage equation obtained above, because $V_o = I_oR$.

Give some time to a careful study of the attached Examples 6.1 and 6.2.

Example 6.1

Find v(t) for $t > 0$ in the circuit below:



Step 1: Obtain the steady-state conditions for $t = 0^-$

At $t=0^-$, by voltage division $v = \frac{10}{25} \times 100 = 40V$ At $t=0^+$, $v = 40V$ by capacitor action

At $t=0^+$

Step 2: Obtain the time constant for the RC section after $t = 0$

$\tau = RC = 10\Omega \times 1F = 10s$

Step 3: Write down the expression for the decaying exponential

$v(t) = 40e^{-t/\tau}$ V

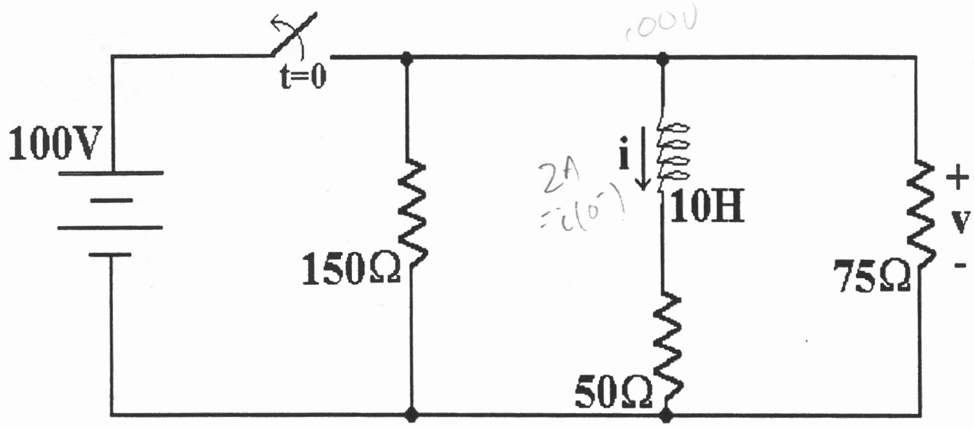
Question 1. Sketch the function $v(t)$
Question 2. What is $v_1(t)$ for the same circuit?

by voltage division

$v = \frac{2}{10} \times v(t) = 8e^{-t/\tau}$

Example 6.2

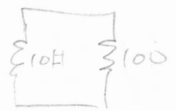
Find the current $i(t)$ for $t > 0$ in the circuit below:



Step 1: Obtain the dc steady-state conditions for $t = 0^-$ (inductor is short circuit)

$i(0^+) = 2A$ by inductor action

$t = 0^+$



Step 2: Obtain the time constant for the LR section after $t = 0$

$\tau = \frac{R}{L} = \frac{100}{10} = 10s$

$\frac{L}{R} = \frac{10}{100} = 0.1$ 100ms

Step 3: Write down the function for the decaying exponential

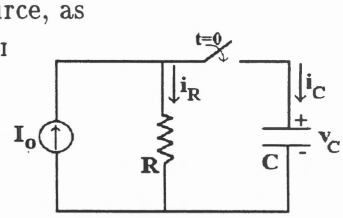
$i(t) = I_0 e^{-t/\tau} = 2 e^{-t/\tau}$

Question. Can you now find $v_R(t)$?

6.4 Response to a Constant Forcing Function

So far there have been no independent sources in the circuits that we have considered, and so we have always been calculating the *natural response*. If, in addition to stored energies, a circuit is driven by independent voltage and/or current sources, then these will produce a *forced response* in a given circuit output, as well as the natural response. We begin by considering circuits including *dc* voltage or current sources only: for such circuits, we say that the *forcing function* is constant.

Suppose we drive an RC combination with a dc current source, as shown in this figure. At $t = 0^-$, let us say that the capacitor voltage is V_o. We want an expression for $v_c(t)$ for $t > 0$ i.e. after the switch has been closed. Applying KCL gives:



$$C\frac{dv}{dt} + \frac{v}{R} - I_o = 0 \qquad \text{so} \qquad \frac{dv}{dt} + \frac{1}{RC}v = \frac{I_o}{C}$$

Separating the variables, we now get:

$$\frac{dv}{dt} = -\frac{v - RI_o}{RC} \qquad \text{so} \qquad \frac{dv}{v - RI_o} = -\frac{1}{RC}dt$$

and then integrating throughout gives us:

$\ln|v - RI_o| = -\dfrac{t}{RC} + K$ so $v - RI_o = e^{-t/(RC)+K}$ so $v = Ae^{-t/(RC)} + RI_o$,

where $A = e^K$ is a constant yet to be determined.

It is worth noticing that this response is in two parts:

1. A decaying exponential, similar to what we get for the un-driven circuit, which we call the *natural response*, v_n. Because it decays away over time, it is sometimes called a *transient response*.

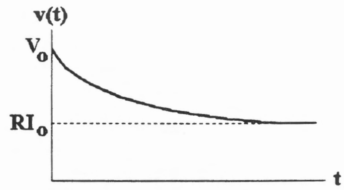
2. A constant, very similar to the forcing function I_o, which dominates the total response after a long enough time, and which we call the *forced response*, v_f. Because of its dominance after a long time, it is sometimes called a *steady state response*.

We still need to find the value of the constant A. Recalling that at $t = 0^-$ $v_c = V_o$, we see that $v_c = V_o$ at $t = 0^+$ also, since the capacitor voltage

cannot change instantaneously. Feeding $t = 0$ and $v = V_o$ into our most recent equation now gives $V_o = A + RI_o$, so $A = V_o - RI_o$. Substituting back, we now finally have the expression for the voltage across the capacitor:

$$v(t) = RI_o + (V_o - RI_o)e^{-t/RC}$$

This function is called the *complete response*, and is sketched in this figure. Note that it is now a simple matter to obtain the current in the capacitor, by writing:



$$i_c(t) = C\frac{dv_c}{dt} = -\frac{V_o - RI_o}{R}e^{-t/RC}$$

You should not have been tempted to use Ohm's Law, which of course only applies for resistors! The resistor current, however, is found that way, and comes out as

$$i_R(t) = I_o + \frac{V_o - RI_o}{R}e^{-t/RC}.$$

Try to sketch $i_c(t)$ and $i_R(t)$, and you will find that $i_c(t) + i_R(t) = I_o$, in accordance with KCL.

6.5 Response to a General Forcing Function

The differential equations that we have so far solved have all been derived from circuits containing only one energy storage element, and have all been of the form

$$\frac{dy}{dt} + Py = Q$$

where P has been a constant. In the source-free circuits, we had $Q = 0$, and in the constant-forcing-function case, Q was a non-zero constant. In general, of course, Q need not be constant, but may be a function of time (e.g. if we drove our circuit with a sinusoidal function). P may also be a function of time, but will never be so in this course.

In order to solve general differential equations of this type, and hence to obtain the response of RC or RL circuits to a forcing function that varies with time, we need to use the *integration factor method*, which is sketched out below:

Multiply both sides of the differential equation by the *integration factor*, $e^{\int Pdt}$, to obtain:

$$\frac{dy}{dt}e^{\int Pdt} + yPe^{\int Pdt} = Qe^{\int Pdt}$$

The LHS of this equation is *exact*, i.e. it is of the form $udv/dt + vdu/dt$, so it is easy to integrate, by applying in reverse the product rule for differentiation. Integrating the last equation throughout therefore gives:

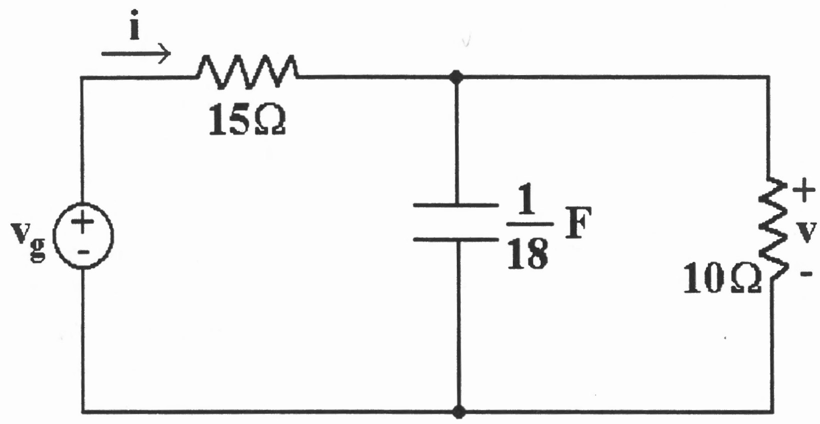
$$ye^{\int Pdt} = \int Qe^{\int Pdt} + K \qquad \text{so} \qquad y = e^{-\int Pdt}\int Qe^{\int Pdt} + Ke^{-\int Pdt}.$$

If P is a constant, then it becomes quite easy to calculate y from here, but remember that if Q is a function of time, then you will have to do some work to integrate the product $Qe^{\int Pdt}$. If both P and Q are constant, then one need not use this method, and should instead look to the methods discussed in the previous sections.

You should make sure that you understand Example 6.3 attached to these notes before attempting Worksheet 6. The methods covered in this section may appear daunting at first, but they become much easier with some regular practice. So far, of course, we are only talking about circuits with *one* energy storage element, which give rise to *first order* differential equations. After practice with analysing these circuits, we will soon take up the challenge of *second order* circuits, in which *two* energy storage elements are present.

Example 6.3

Find v(t) for $t > 0$, if $i(0) = 1\text{A}$ and $v_g = 30e^{-5t}\text{V}$.



Step 1: Obtain the differential equation for v

By KCL $\quad \frac{v - v_g}{15} + \frac{1}{18}\frac{dv}{dt} + \frac{v}{10} = 0$

$12v - 12v_g + 10\frac{dv}{dt} + 18v = 0$

$\frac{dv}{dt} + 3v = 1.2v_g = 36e^{-5t}$

Step 2: Solve the equation, using the appropriate technique Use an integrating factor

$e^{\int 3 \cdot dt} = e^{3t}$ (integrating factor)

$e^{3t}\frac{dv}{dt} + 3e^{3t} = 36e^{-5t}e^{3t} = 36e^{-2t}$ Integrate

$ve^{3t} = 36\int e^{-2t}\,dt = -18e^{-2t} + k$

$v = -18e^{-5t} + ke^{-3t}$

Step 3: Find the initial conditions, and apply them.

$V_0 = V_g(0) - 15\,i(0)$

$= 30 - 15 = 15V$

$15 = -18 + k \quad \therefore k = 33$

$v(t) = 33e^{-3t} - 18e^{-5t}$ V

TEST: Don't be surprised if you have to discuss in words ie in 100-200 words, explain

Let's do drawing work – use rulers, pencils, markers, labels, be neat

TUTORIAL 6

6.1

(a) In Figure 6.1(a), let $t_o = 0$, $V_o = 10$V, $R = 1k\Omega$ and $C = 1\mu F$. Find v, i and w_c at $t = 1$ms.
(3.68V, 3.68mA, 6.8μJ)
(b) Given that the circuit of Figure 6.1(b) is in dc steady state at $t = 0^-$, find v(t) for $t > 0$.
($25e^{-2t}V$)

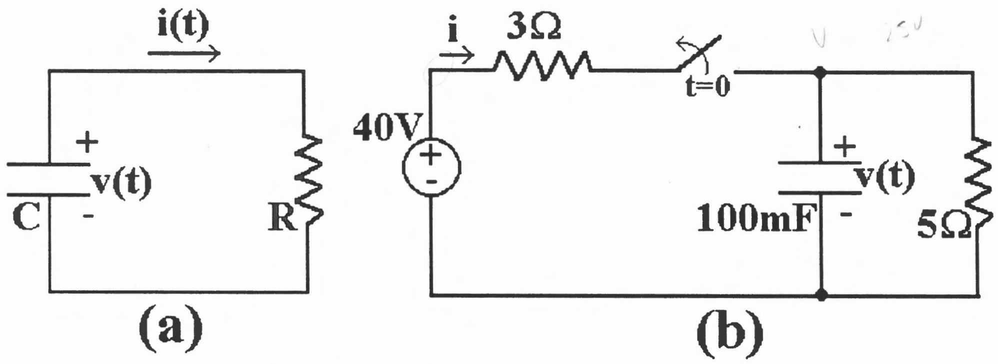


Figure 6.1: Figure for Question 6.1

6.2

If the circuit of Figure 6.2 is in steady state at $t = 0^-$, find i(t) for $t > 0$.
($2e^{-2t}$A)

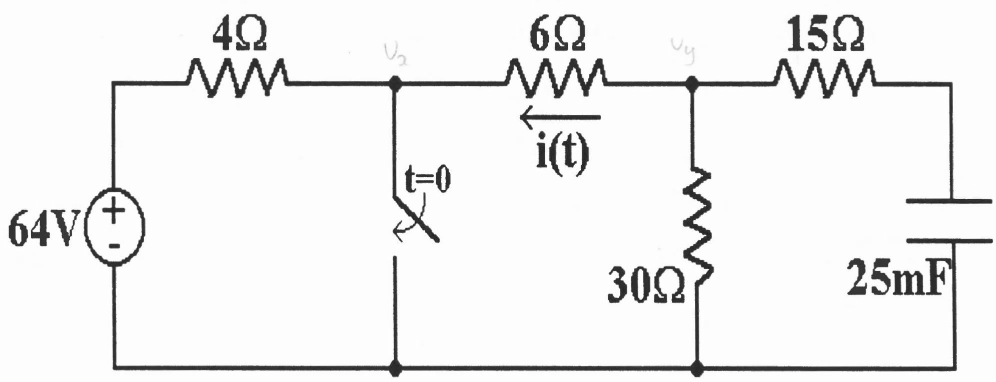


Figure 6.2: Figure for Question 6.2

$v_2 = \frac{36}{40}\times 64 = 57.6V$ $V_y = \frac{30}{40}\times 64 = 48V$

$i(0^-) = \frac{48-57.6}{6\Omega} = -1.6A$

$v(0^-) = V_y = 48V = v(0^+)$ by capacitor action

$r = 15 + 30//6 = 20\Omega$; $RC = 20\times 0.025 = \frac{1}{2}$ s

$v(t) = 48e^{-2t}$ by voltage division, $v_y = \frac{5}{5+15}\times v(t) = 12e^{-2t}$

$\therefore i = \frac{v(t)}{6} = 2e^{-2t}$

6.3

The circuit shown in Figure 6.3 is in dc steady state at $t = 0^-$, and the switch is moved from position 1 to position 2 at $t = 0$. Find v(t) for $t > 0$. ($8e^{-3t}$V)

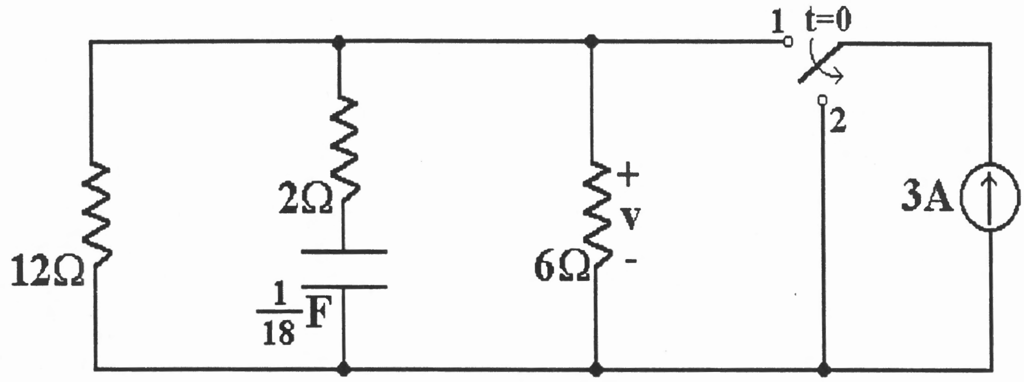


Figure 6.3: Figure for Question 6.3

6.4

Find i(t) for $t > 0$ if the circuit of Figure 6.4 is in steady state at $t = 0$. ($250e^{-2t}$mA)

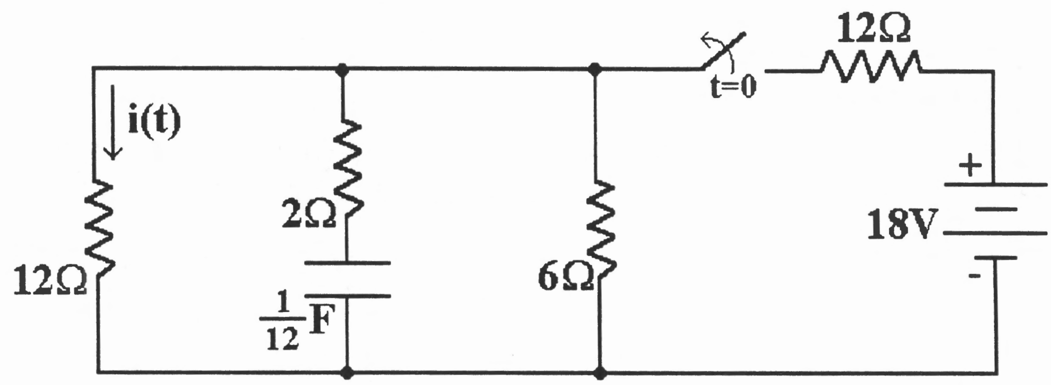


Figure 6.4: Figure for Question 6.4

6.5

The circuit of Figure 6.5 is in dc steady state at $t = 0^-$. Find expressions for i(t) and v(t) for $t > 0$.
$(3e^{-2t}\text{A}, -6e^{-2t}\text{V})$

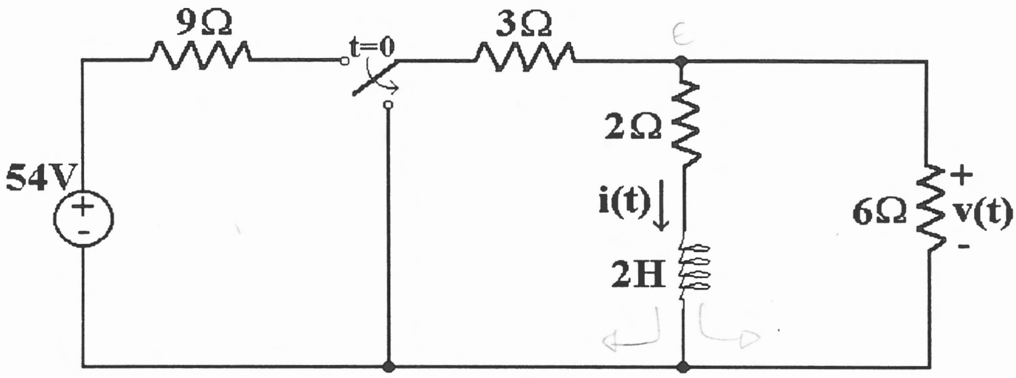


Figure 6.5: Figure for Question 6.5

6.6

Find v(t) for $t > 0$ if the circuit of Figure 6.6 is in steady state at $t = 0^-$. Use "separation of variables" in your working and sketch your final answer.
$(10 - 6e^{-50t}\text{V})$

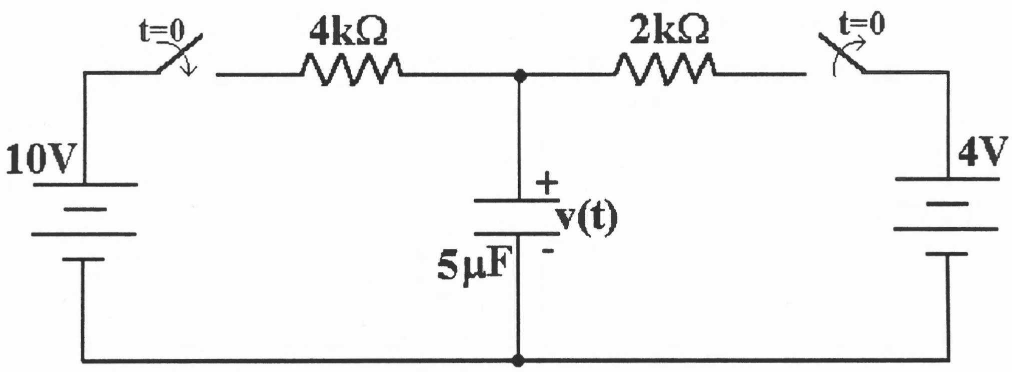


Figure 6.6: Figure for Question 6.6

At $t=0^-$, $i = \frac{24}{8+4} = 2A$

$t=0^+$ $V = 24 - 8.2A = 24 - 16 = 8$

$\frac{24-V}{8} - \frac{1}{2}\int V\,dt = 0$

$24 - V - 4\int V\,dt = 0$

$\frac{dV}{dt} + 4V = 0$

$sAe^{st} + 4Ae^{st} = 0$

$Ae^{st}(s+4) = 0$

$\therefore s = -4$

At $t=0^+$, $Ae^{st} = 8$ $\therefore A = 8$

$\therefore V(t) = \cancel{Ae^{-4t}}\ 8e^{-4t}$

$i(t) = \frac{24 - V(t)}{8} = 3 - e^{-4t}$

6.7

The circuit of Figure 6.7 is in dc steady state at $t = 0^-$. Find i(t) for $t > 0$, using the "s-method" in your working. Sketch your answer.
$(3 - e^{-4t}\text{A})$ to use s-method, you must use KCL, and integrate. See left

$V(t) = Ae^{st}$

$\frac{dV}{dt} = sAe^{st}$

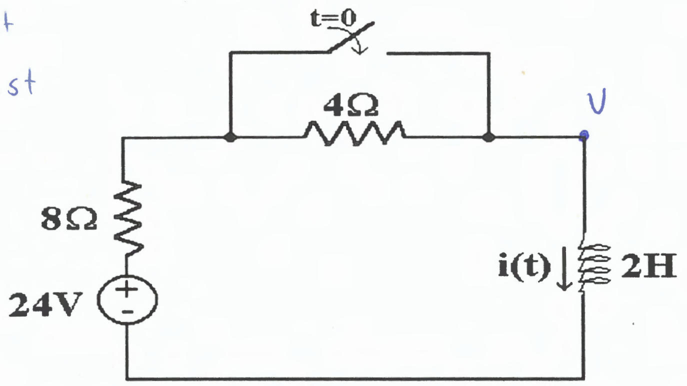


Figure 6.7: Figure for Question 6.7

6.8

Find and sketch v(t) for $t > 0$ if the circuit of Figure 6.8 is in dc steady state at $t = 0^-$. Use either "separation of variables" or the "s-method" in your working, but do *not* use an integrating factor.
$(24 - 8e^{-3t}\text{V})$

16V = v(0⁺)

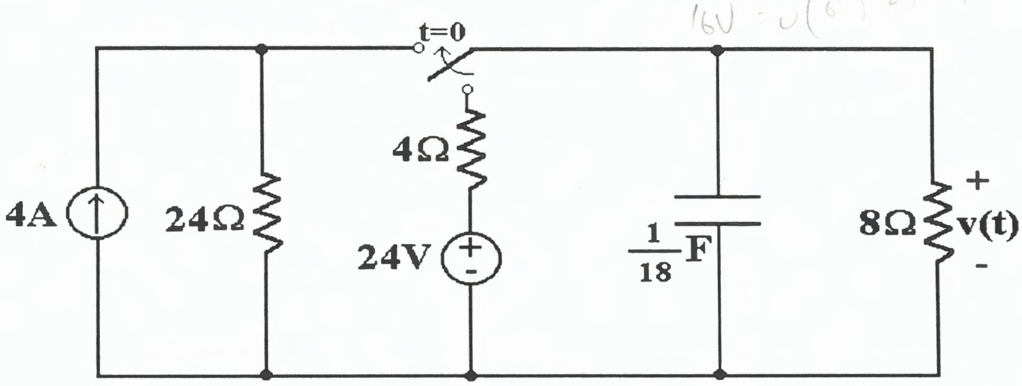


Figure 6.8: Figure for Question 6.8

only finding natural response

$-4 + \frac{V}{24} + \frac{1}{18}\frac{dV}{dt} + \frac{V}{8}$

$\frac{1}{18}\frac{dV}{dt} + \frac{V}{6} = 4$

$\frac{dV}{dt} + 3V = 72$

$V = V_f + V_n$

Let $V = Ae^{st}$

$sAe^{st} + 3Ae^{st} = \cancel{72}\ 0$

Then $Ae^{st}(s+3) = 0$

$\therefore s = -3$

Then $V(t) = \cancel{16}\ A e^{-3t}$

forced response is constant : $V_f = C$. using differential eqn $0 + 3C = 72$

$V_f = 24$

initial condition : $16 = 24 - A$

$\therefore A = -8$

$\therefore V(t) = -8e^{-3t}$

6.9

In the circuit of Figure 6.9, $i(0) = 1\text{A}$. Find and sketch v(t) for $t > 0$ if

(a) $v_g = 50\text{V}$ (use separation of variables)
(b) $v_g = 30e^{-5t}\text{V}$ (use an integrating factor)

$(20 + 15e^{-3t}\text{V};\ 33e^{-3t} - 18e^{-5t}\text{V})$

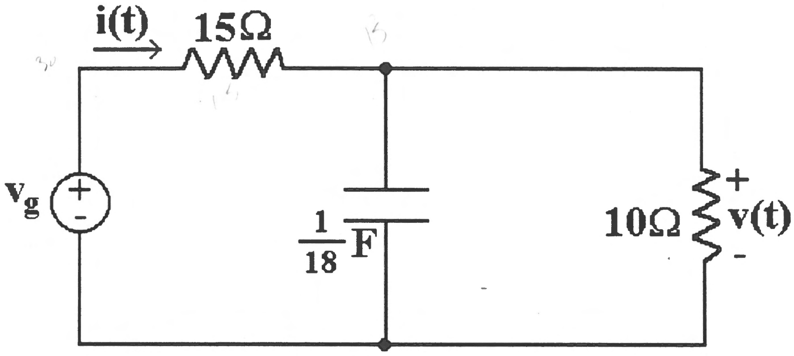


Figure 6.9: Figure for Question 6.9

6.10

In the circuit of Figure 6.10, $i(0) = 4\text{A}$. Find and sketch i(t) for $t > 0$ if

(a) $i_g = 8\text{A}$ (use separation of variables)
(b) $i_g = 13\cos(t)\text{A}$ (use an integrating factor) - Not be asked in test or exam

$(2 + 2e^{-8t}\text{A},\ 400(8\cos(t) + \sin(t) + 2e^{-8t})\text{mA})$

Stopper question

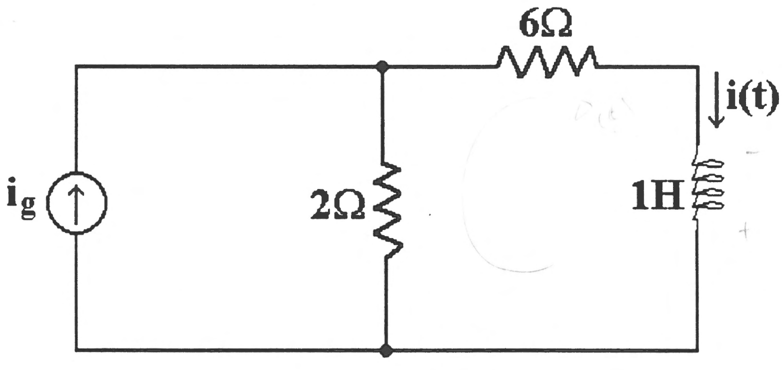


Figure 6.10: Figure for Question 6.10

WORKSHEET 6

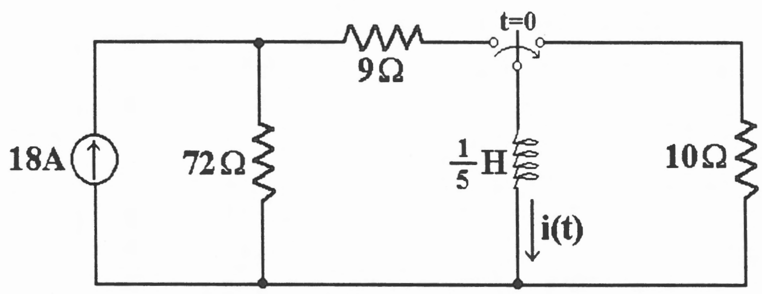
6.A*

(a) The circuit shown below is in the dc steady state at $t = 0^-$.

(i) Find a simple expression for $i(t)$ for $t > 0$. (There is no need to write down or to solve any differential equation explicitly, although you may do so if you wish).

(ii) Sketch the graph of the function found in part (i), and identify the time constant on your sketch. State its value.

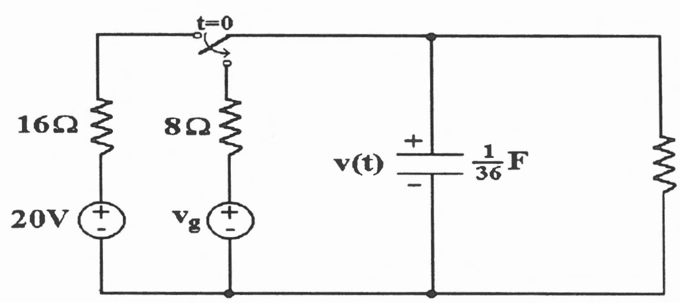
(iii) After how many time constants has any decaying exponential fallen to **exactly 1%** of its original value?



(b) Find $v(t)$ for $t > 0$ by setting up a first order differential equation in standard form, and then solving it using the method indicated. The circuit below is in the dc steady state at $t = 0^-$ and

(i) $v_g = 8$ V (use **separation of variables**)

(ii) $v_g = 6e^{-3t}$V (use an **integrating factor**)



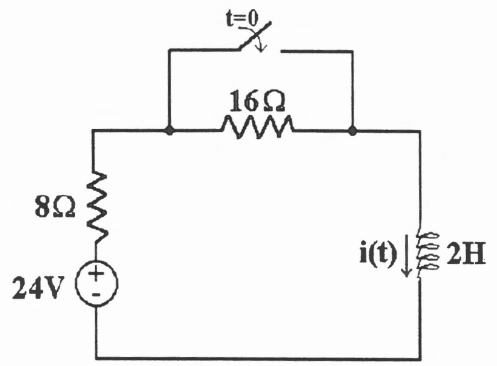
6.B

The circuit in the Figure below is in the dc steady state at $t = 0^-$.

(a) Derive a differential equation for the current, in standard form.

(b) Find $i(t)$ for $t > 0$, using any appropriate method to solve the differential equation for the current.

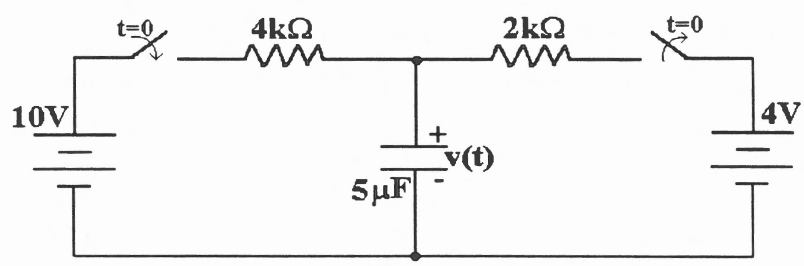
(c) Make a sketch of the current as a function of time.



6.C

Find $v(t)$ for $t > 0$ in the circuit shown below, if the circuit is in the dc steady state at $t = 0^-$. Give the following in your answer:

(a) The initial condition, $v(0)$;
(b) The differential equation for $v(t)$, in standard form;
(c) The solution for $v(t)$, using "separation of variables";
(d) A sketch of v(t), including the time constant.



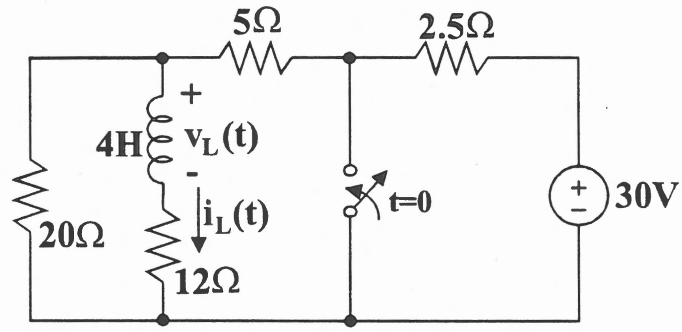
6.D*

(a) The circuit shown below is in the dc steady state at $t = 0^-$.

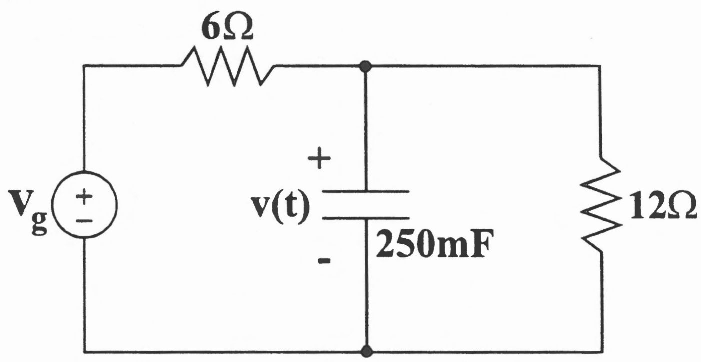
(i) Find a simple expression for $i_L(t)$ for $t > 0$. (There is no need to write down or to solve any differential equation explicitly, although you may do so if you wish).

(ii) Sketch the graph of the function found in part (i), and identify the time constant on your sketch. State its value.

(iii) Now find an expression for $v_L(t)$ for $t > 0$.



(b) Find $v(t)$ for $t > 0$ in the circuit below by setting up a first order differential equation in standard form, and then solving it using an appropriate method. You are given that $v(0) = -6\text{V}$, and that $v_g = 36e^{-3t}\text{V}$.



6.E

The circuit in the figure below is in the dc steady state at $t = 0^-$, and at $t = 0$ the two switches are flipped as shown. Find $v(t)$ for $t > 0$. Your answer should proceed as follows:

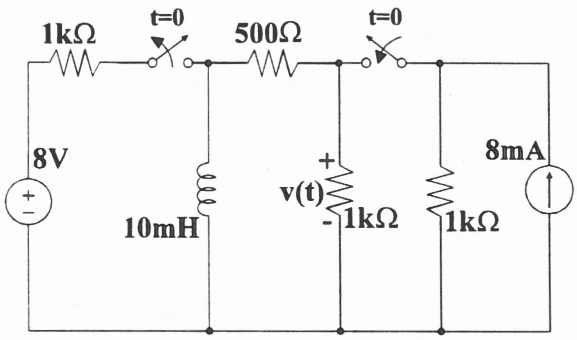
(a) Find the initial condition, $v(0)$, explaining your reasoning carefully;

(b) Show that the differential equation for $v(t)$ in standard form is

$$\frac{dv}{dt} + 10^5 v = 2 \times 10^5$$

[4]

(c) Solve this equation, using an appropriate method.



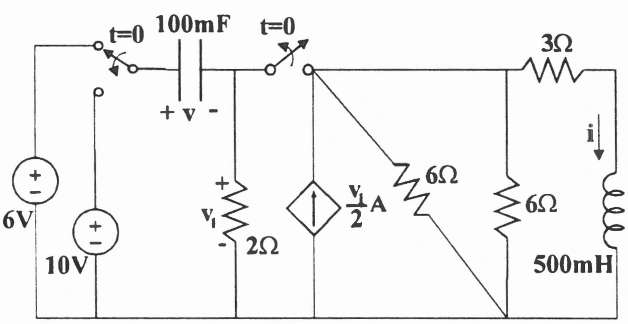
6.F

The circuit below is in the dc steady state at time $t = 0^-$.

(a) Find v, v_1 and i at $t = 0^-$ and at $t = 0^+$.

(b) Find $\frac{dv}{dt}$ and $\frac{di}{dt}$ at $t = 0^+$.

(c) Calculate $v(t)$ for $t > 0$.



Chapter 7

Second Order Circuit Analysis

In Lectures C13, C14 and C15 we cover:

- Circuits containing *two* energy storage elements (e.g. CC, CL, LL), which are usually described by *second order* differential equations
- How to find the *natural response* of a second order circuit, using the "s-method", which we touched on in studying first-order circuits
- How to find the *forced response* of a second order circuit, by means of assuming solutions called *trial solutions* in advance
- Two very important second order circuits, known as the *parallel RLC circuit* and the *series RLC circuit*

7.1 Circuits with Two Storage Elements

Circuits that contain two energy storage elements may sometimes be described by the *first* order differential equations which are now familiar to us. In the circuit of Figure 7.1(a), for example, nodal equations at v_1 and v_2 give (by KCL):

$$\frac{v_1 - v_g}{1} + \frac{dv_1}{dt} = 0, \qquad \text{so } \frac{dv_1}{dt} + v_1 = v_g,$$

and

$$\frac{v_2 - v_g}{2} + \frac{1}{4}\frac{dv_2}{dt} = 0, \qquad \text{so } \frac{dv_2}{dt} + 2v_2 = 2v_g$$

Both of these equations are in the first order *standard form.* To find either v_1 or v_2, all we have to do is to *solve* the appropriate differential equation, using one of the techniques that we have studied (separation of variables, the "s-method" or by using an integrating factor). Sometimes, however, the

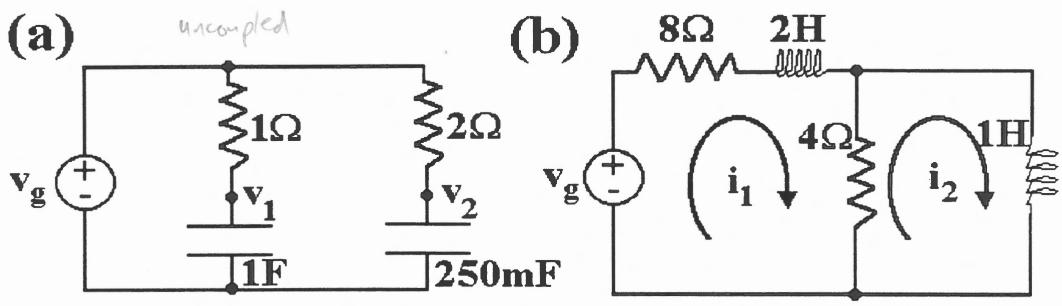


Figure 7.1: (a) A First Order Circuit (b) A Second Order Circuit

differential equations describing a circuit are *coupled*, as in the example of Figure 7.1(b). Here, the mesh equations give (by KVL):

$$v_g - 8i_1 - 2\frac{di_1}{dt} - 4(i_1 - i_2) = 0 \qquad \text{so } 2\frac{di_1}{dt} + 12i_1 - 4i_2 = v_g \text{ (Eq 1)}$$

$$\text{and } -4(i_2 - i_1) - 1\frac{di_2}{dt} = 0 \qquad \text{so } \frac{di_2}{dt} + 4i_2 - 4i_1 = 0 \text{ (Eq 2)}$$

You can *decouple* the equations like this:

From Eq 2:
$$i_1 = \frac{1}{4}\left(\frac{di_2}{dt} + 4i_2\right)$$

So in Eq 1 this gives
$$\frac{2}{4}\left(\frac{d^2i_2}{dt^2} + 4\frac{di_2}{dt}\right) + \frac{12}{4}\left(\frac{di_2}{dt} + 4i_2\right) - 4i_2 = v_g,$$

or, simplifying,
$$\frac{1}{2}\frac{d^2i_2}{dt^2} + 5\frac{di_2}{dt} + 8i_2 = v_g,$$

or
$$\frac{d^2i_2}{dt^2} + 10\frac{di_2}{dt} + 16i_2 = 2v_g,$$

which is a *second order* differential equation in *standard form.*

We aim to *solve* this equation, which means finding $i_2(t)$. Note that the RHS contains the *forcing function* v_g, while the solution to the equation "LHS=0" would be the *natural response*, just as we had in the first order

case. Note also that our final solution will be the *complete response*, which is made up of the natural and forced responses like this:

$$i_{2c}(t) = i_{2n}(t) + i_{2f}(t)$$

Now the natural response satisfies

$$\frac{d^2 i_{2n}}{dt} + 10\frac{di_{2n}}{dt} + 16i_{2n} = 0$$

while the forced response will satisfy

$$\frac{d^2 i_{2f}}{dt} + 10\frac{di_{2f}}{dt} + 16i_{2f} = 2v_g$$

Adding these equations, we get

$$\frac{d^2}{dt^2}(i_{2n} + i_{2f}) + 10\frac{d}{dt}(i_{2n} + i_{2f}) + 16(i_{2n} + i_{2f}) = 2v_g$$

or
$$\frac{d^2 i_{2c}}{dt} + 10\frac{di_{2c}}{dt} + 16i_{2c} = 2v_g,$$

showing that the complete response is indeed the sum of the natural and forced responses that we have defined. All of this is valid because the equations involved are *linear*.

This strongly suggests that the way to find the complete response is to split up the problem. We will now firstly study how to get the natural response, and then move on to get the forced response, and finally we add them to get the complete solution.

7.2 The Natural Response

The natural response is the solution of the equation that results from setting the original LHS=0. This is called the *homogeneous equation*, and in our case it is

$$\frac{d^2 i_2}{dt^2} + 10\frac{di_2}{dt} + 16i_2 = 0$$

The only way that a function $i_2(t)$ could add up with its derivatives $\frac{di_2}{dt}$and $\frac{d^2 i_2}{dt^2}$ in any linear combination and somehow give zero (as this equation implies) would be if $i_2(t)$ were of the form $i_2(t) = Ae^{st}$. Think about this, and you will see that there is no true counter-example to it. This means

that we can solve the homogeneous equation using the "s-method". Upon feeding our supposed solution into the equation we get

Natural frequencies [measured in s^{-1}]

$$As^2e^{st} + 10Ase^{st} + 16Ae^{st} = 0$$

so

$$Ae^{st}(s^2 + 10s + 16) = 0$$

characteristic equation

As in the first-order case, this only leaves the following ways to proceed:

- $A = 0$, which is the *trivial case*, where $i_2(t) = 0$, which quite clearly gets us nowhere!
- $e^{st} = 0$, which is not possible...
- $s^2 + 10s + 16 = 0$, which gives $s = -2$ or $s = -8$

The third approach above has therefore given us two possible solutions to the homogeneous equation: either $i_{2n}(t) = A_1e^{-2t}$ or $i_{2n}(t) = A_2e^{-8t}$. You can check that either of these satisfies the homogeneous equation by substituting it back in; but check further and you will verify that (because the equations are linear) the *sum* of these solutions also satisfies the equation, and we adopt this as the most general solution available. Hence

$$i_{2n}(t) = A_1e^{-2t} + A_2e^{-8t}$$

The equation $s^2 + 10s + 16 = 0$ is called the *characteristic equation* of the circuit, and its solutions $s_1 = -2$ and $s_2 = -8$ are sometimes known as the *natural frequencies* of the circuit. Keep an eye open for more news of these in later courses.

7.3 Types of Natural Frequencies

You can see that in general a second-order circuit will have a characteristic equation which is *quadratic in s*. There is nothing to say that the roots of this equation will be two different numbers, or even two *real* numbers, as they were in our example above. There are actually three possibilities, and here *without proof* is what to do in each case. (Mathematics courses on differential equations will go into this in much more detail).

- CASE 1: the roots are *real* and *distinct* (like s_1 and s_2 in our example). In this situation, choose the solution to the homogeneous equation to be

$$y_n t = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

just as we did above. You will find that s_1 and s_2 will be negative, so that if you sketch the solution as a function of t, it will eventually die away. For this reason, we know this as the *overdamped case.*

- CASE 2: the roots are *real*, but *equal* (i.e. $s_1 = s_2 = s$). This doesn't happen very often, but if it does you choose the solution to the homogeneous equation to be

$$y_n(t) = (A_1 + A_2 t)e^{st}$$

This is known as the *critically damped* case, and it is of considerable theoretical interest

- CASE 3: the roots are *complex numbers* in which case they will be a *complex conjugate pair* (i.e. $s_{1,2} = \alpha \pm j\beta$, where $j = \sqrt{-1}$). If this happens, then choose then solution to the homogeneous equation to be

$$y_n(t) = e^{\alpha t}(A_1 \cos(\beta t) + A_2 \sin(\beta t))$$

You can probably see that, over time, this function *oscillates* and then dies away (provided that α is negative). It is a very interesting and frequently-encountered situation known as the *underdamped case*

To summarise the position so far, we have obtained a second order differential equation for a current or a voltage in our circuit. We can find its natural response simply by applying the "s-method" to the standard form of the differential equation, solving the characteristic equation and then writing down the homogeneous solution, depending on the case given by examining s_1 and s_2. Each natural solution has two constants (A_1 and A_2), to find which we would need two further pieces of information, known as *boundary* or *initial conditions.* We shall see later how this is done.

7.4 The Forced Response

We are now half-way to getting the complete response of the circuit! The original second order differential equation will generally have a non-zero RHS, which is the driving or forcing function. We must find a final solution which takes this into account *as well as* the natural response of the circuit.

Once again, this will be covered in detail in maths courses, so for the moment we will simply observe that in first-order systems we noted that the forced response always "looks like" the forcing function. The same is true in the second order systems that we deal with. To find the forced response, we *assume* that it will take the form of the RHS, and select a *trial solution* from the table below: Hence, in our original example, if $v_g = 16$, then we

RHS of DE	Trial Solution
K (a constant)	A (a constant)
$at + K$	$At + B$
$at^2 + bt + K$	$At^2 + Bt + C$
e^{at}	Ae^{at}
$\sin(bt)$, $\cos(bt)$	$A\cos(bt) + B\sin(bt)$
$e^{at}\sin(bt)$, $e^{at}\cos(bt)$	$e^{at}(A\cos(bt) + B\sin(bt))$

begin with

$$\frac{d^2 i_2}{dt^2} + 10\frac{di_2}{dt} + 16i_2 = 2v_g = 32$$

so

$$\frac{d^2 i_{2f}}{dt^2} + 10\frac{di_{2f}}{dt} + 16i_{2f} = 32$$

has a *constant* RHS. We therefore assume that the forced response is also a constant (i.e. $i_{2f} = A$). Feeding this back into the DE, we get

$$0 + 0 + 16A = 32$$

and so we find simply that $A = 2$. Hence $i_{2f} = 2$

The *complete response* is now finally obtained by adding the natural and forced components, to get

$$i_{2c}(t) = i_{2n}(t) + i_{2f}(t) = A_1 e^{-2t} + A_2 e^{-8t} + 2$$

Knowledge of the initial energies stored in the inductors might now be used to find A_1 and A_2. We will see how in a later example.

Suppose that in our original example we had had a sinusoidal input, with $v_g = 20\cos(4t)$V. Let us see how this would have changed the outcome. In this event,

$$\frac{d^2 i_{2f}}{dt^2} + 10\frac{di_{2f}}{dt} + 16 i_{2f} = 2v_g = 40\cos(4t)$$

We try the solution $i_{2f}(t) = A\cos(4t) + B\sin(4t)$ suggested in the table of trial solutions. Since:

$$\frac{di_{2f}}{dt} = -4A\sin(4t) + 4B\cos(4t) \quad \text{and} \quad \frac{d^2 i_{2f}}{dt^2} = -16A\cos(4t) - 16B\sin(4t)$$

substitution back into the original equation now gives us:

$$40B\cos(4t) - 40A\sin(4t) = 40\cos(4t)$$

Equating sine and cosine terms, it is now apparent that $40B = 40$, so $B = 1$ and $40A = 0$, so $A = 0$. Hence, our forced response is simply $i_{2f} = \sin(4t)$, and therefore the complete response is

$$i_{2c}(t) = i_{2n}(t) + i_{2f}(t) = A_1 e^{-2t} + A_2 e^{-8t} + \sin(4t)$$

You could check this by resubstitution into the original equation, or print it out on a computer screen to see what it looks like. As before, A_1 and A_2 would only be found if there were extra information available. You will see how this is done in the following examples.

Example 7.1

Find $x(t)$ for $t > 0$, which satisfies the system of equations:

$$\frac{dx}{dt} + 2x + 5\int_0^t x\,dt = 16e^{-3t}$$

$$x(0) = 2$$

Step 1: Obtain the DE in standard form.

Differentiate eqn: $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = -48e^{-3t}$

From fundamental th. of calculus

Step 2: Extract and solve the characteristic equation.

$s^2 + 2s + 5 = 0$ (for natural response)

$s = \frac{-2 \pm \sqrt{4-20}}{2}$ $\therefore$ $s = -1 \pm j2$

Step 3: Write down the natural response

Natural response: $x_n(t) = e^{\alpha t}(A_1 \cos\beta t + A_2 \sin(\beta t))$

$= e^{-t}(A_1 \cos(2t) + A_2 \sin(2t))$

Step 4: Find the forced response, using a trial solution.

As forced function is $-48e^{-3t}$, trial solution is $x_f(t) = Ae^{at}$

$x_f(t) = Ae^{-3t}$ $\quad x_f'(t) = -3Ae^{-3t}$ $\quad x_f''(t) = 9Ae^{-3t}$

Back into the d.e.: $9Ae^{-3t} - 6Ae^{-3t} + 5Ae^{-3t} = -48e^{-3t}$ $\quad 8A = -48$ $\quad \therefore A = -6$ $\quad x_f(t) = -6e^{-3t}$

Step 5: Write down the complete response

$x_c(t) = x_n(t) + x_f(t)$

$= e^{-t}(A_1\cos(2t) + A_2\sin(2t)) - 6e^{-3t}$

Step 6: Find two initial conditions, and use them to complete the solution.

$x(0) = 2$: (Sub into eqn above)

$\therefore$ $A_1 - 6 = 2$ $\quad \therefore A_1 = 8$

Using first eqn, when $t = 0$:

$\frac{dx(0)}{dt} + 2x(0) + 5\int_0^{t=0} x\,dt = 16e^{-3(0)}$

$\therefore \frac{dx(0)}{dt} = 16 - 4 = 12$

differentiate complete response, and put $t = 0$

$2A_2 - A_1 + 18 = 12$

$2A_2 - 8 + 18 = 12$

$A_2 = 1$

$\therefore x_c(t) = e^{-t}(8\cos(2t) + \sin(2t)) - 6e^{-3t}$

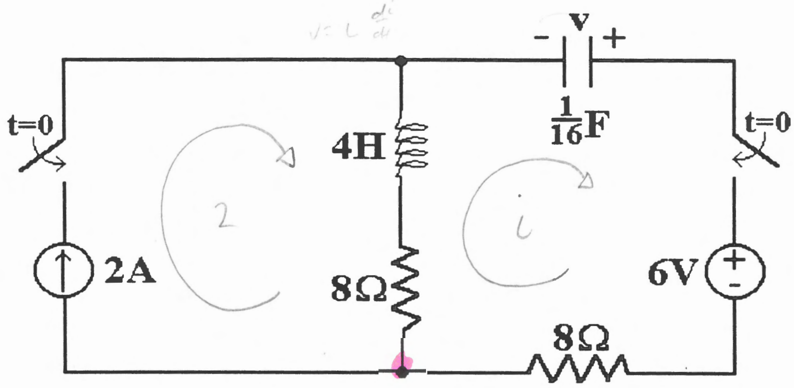
Example 7.2

Find $v(t)$ for $t > 0$, if the circuit is in dc steady state at $t = 0^-$.

At $t = 0^-$

$i(0^-) = 0 = i(0^+)$ by inductor action

$v(0^-) = 0 = v(0^+)$ by capacitor action



By KCL and inductor action, that at $t=0$, the 2A goes through the capacitor (from node marked)

$$2A = -\frac{1}{16}\frac{dv(0)}{dt} \qquad \frac{dv_0}{dt} = -32$$

Using mesh analysis: $-8(i-2) - 4\frac{d}{dt}(i-2) + v - 6 - 8i = 0$

so $4\frac{di}{dt} + 16i - 10 = v$ but $i = C\frac{dv}{dt} = -\frac{1}{16}\frac{dv}{dt}$

so $-\frac{4}{16}\frac{d^2v}{dt^2} - \frac{16}{16}\frac{dv}{dt} - 10 = v$

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 4v = -40$$ Standard form

Characteristic equ: $s^2 + 4s + 4 = 0$

$(s+2)^2 = 0$ ∴ $s = -2$ roots are real, equal

trial solution: $v_n(t) = (A_1 + tA_2)e^{-2t}$

For forced response: RHS $= -40$ (constant)

trial solution: $v_f(t) = A$

Put into DE: $0 + 0(4) + 4A = -40$

∴ $A = -10$

Complete solution: $v_c(t) = (A_1 + tA_2)e^{-2t} - 10$

initial conditions on v: $v(0) = 0$: $A_1 - 10 = 0$ ∴ $A_1 = 10$

$\frac{dv(0)}{dt} = -32$: $-2A_1 + A_2 = -32$

$-20 + A_2 = -32$ ∴ $A_2 = -12$

∴ $v(t) = (10 - 12t)e^{-2t} - 10$ V

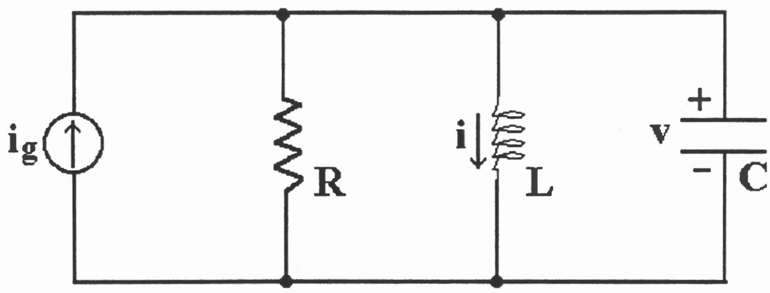


Figure 7.2: A Parallel RLC Circuit

7.5 The Parallel RLC Circuit

A very important second-order circuit is the *parallel RLC circuit* shown in Figure 7.2. Suppose that the RLC section is driven by the current source i_g, and that at $t = 0$, the initial conditions in the circuit are $i(0) = I_o$ and $v(0) = V_o$. We are now in a position to analyse the circuit algebraically.

To begin with, nodal analysis gives:

$$\frac{v}{R} + \frac{1}{L}\int_0^t v\,dt + I_o + C\frac{dv}{dt} = i_g$$

and so, differentiating this *integro-differential equation*, we get:

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = \frac{di_g}{dt}$$

The natural response is found from

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0,$$

which yields the characteristic equation:

$$Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0,$$

from which $s = \dfrac{-\frac{1}{R} \pm \sqrt{\frac{1}{R^2} - \frac{4C}{L}}}{2C} \quad = \quad -\dfrac{1}{2RC} \pm \sqrt{\left(\dfrac{1}{2RC}\right)^2 - \dfrac{1}{LC}}$

Clearly, the type of response the circuit gives will depend on which case we have here, and that depends on the square-rooted quantity in the expression above, which is known as the *discriminant.*

- If $\frac{1}{R^2} - \frac{4C}{L} > 0$ (i.e. $L > 4R^2C$), then the natural frequencies are real and distinct (and negative), so we have the overdamped case and

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- If $\frac{1}{R^2} - \frac{4C}{L} = 0$ (i.e. $L = 4R^2C$), then the natural frequencies are real and equal (and negative), so we have the critically-damped case and

$$v(t) = (A_1 + A_2 t)e^{st}$$

- If $\frac{1}{R^2} - \frac{4C}{L} < 0$ (i.e. $L < 4R^2C$), then the natural frequencies are *complex* and we have the underdamped case with v(t) containing sinusoidal functions. This implies that the circuit will respond in an *oscillatory* fashion (i.e. it will *resonate*). Such a result may be highly desirable (e.g. in a tuning circuit in a radio), or possibly very undesirable (e.g. where you want the voltage to "settle down" as fast as possible such as in the circuit that supplies power to the radio!).

In the third case above, since $s = \alpha \pm j\beta$, you can see that

$$\alpha = -\frac{1}{2RC} \qquad \text{and} \qquad j\beta = \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

The response is then

$$e^{\alpha t}(A_1 \cos(\beta t) + A_2 \sin(\beta t)).$$

Think about sketching this function. Evidently α determines how fast it dies away, and is therefore called the *damping coefficient.* β is called the *damped frequency*, but, more importantly, notice how $\beta = \sqrt{\omega_o^2 - \alpha^2}$, where the quantity $\omega_o = \frac{1}{\sqrt{LC}}$ is an extremely useful parameter (known as the *resonant frequency*) that tells at what frequency the circuit will continue to oscillate forever, if the resistor was removed (i.e. if there were infinite resistance present). Such oscillation has zero damping coefficient, and may also be highly desirable (e.g. in the crystal oscillator of a computer). Whether it may be achieved in practice will have to wait for a later course.

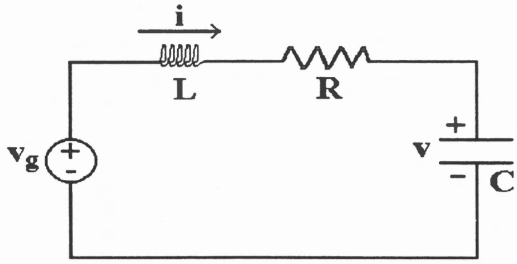


Figure 7.3: A Series RLC Circuit

7.6 The Series RLC Circuit

This is another important circuit, which we can analyse with the tools now at our disposal. A *series RLC* combination is shown in Figure 7.3, driven by a voltage sourcev_g, and, as we did before, we shall assume the initial conditions to be $i(0) = I_o$ and $v(0) = V_o$. In brief, the analysis is shown below.

By KVL, $v_g - L\dfrac{di}{dt} - Ri - \dfrac{1}{C}\displaystyle\int_0^t i\,dt - V_o = 0$ so $L\dfrac{d^2i}{dt^2} + R\dfrac{di}{dt} + \dfrac{1}{C}i = \dfrac{dv_g}{dt}$

The natural response is deduced from the characteristic equation:

$$Ls^2 + Rs + \frac{1}{C} = 0,$$

From this, you can see that the natural frequencies are

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

We can again use the disriminant to observe that:

- If $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$ (i.e. $C > 4\frac{L}{R^2}$), then the circuit is overdamped and $i(t) = A_1e^{s_1t} + A_2e^{s_2t}$
- If $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$ (i.e. $C = 4\frac{L}{R^2}$), then the circuit is critically damped and $i(t) = (A_1 + A_2t)e^{st}$
- If $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$ (i.e. $C < 4\frac{L}{R^2}$), then the circuit is underdamped and it will oscillate with resonant frequency $\omega_o = \frac{1}{\sqrt{LC}}$, damping coefficient $\alpha = -\frac{R}{2L}$ and damped frequency $\beta = \sqrt{\omega_o^2 - \alpha^2}$. The response is $i(t) = e^{\alpha t}(A_1\cos(\beta t) + A_2\sin(\beta t))$

TUTORIAL 7

Exam questions based on this these questions

7.1
(a) Show that

$$x_1 = A_1 e^{-2t}$$

and

$$x_2 = A_2 e^{-3t}$$

are each solutions of

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0,$$

regardless of the values of the constants A_1 and A_2.

(b) Show that

$$x = x_1 + x_2 = A_1 e^{-2t} + A_2 e^{-3t}$$

is also a solution of the equation of part (a).

(c) Show that if the RHS of the differential equation of part (a) is changed from 0 to 12, then

$$x = A_1 e^{-2t} + A_2 e^{-3t} + 2$$

is a solution. Thus, the natural response is $A_1 e^{-2t} + A_2 e^{-3t}$, and the forced response is 2.

7.2

(a) Find the natural frequencies of a circuit described by

$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_o x = 0$$

if

(i) $a_1 = 5$, $a_o = 4$ (ii) $a_1 = 4$, $a_o = 13$ (iii) $a_1 = 8$, $a_o = 16$.

(-1, -4; -2±j3; -4, -4)

(b) Solve for x in the equations of part (a), given that the initial conditions have been determined to be $x(0) = 3$ and $dx(0)/dt = 6$.

$(6e^{-t} - 3e^{-4t};\ e^{-2t}(3\cos(3t) + 4\sin(3t));\ (3 + 18t)e^{-4t})$

7.3

(a) If the node voltage at node b in Figure 7.4 is v_1, write down the two nodal equations for nodes a and b.

$$\frac{V_g - V}{4} - \frac{V - V_1}{R} - \frac{1}{4}\frac{dV}{dt} = 0$$

$$RV_g - RV - 4V + 4V_1 - R\frac{dV}{dt} = 0$$

$$\boxed{RV_g + 4V_1 = R\frac{dV}{dt} + V(R+4)} \quad (1)$$

$$\frac{V - V_1}{R} - \int_0^t V_1 = 0$$

$$\boxed{\frac{dV}{dt} - \frac{dV_1}{dt} = RV_1} \quad (2)$$

$$4\frac{dV}{dt} - 4\frac{dV_1}{dt} = 4RV_1$$

(b) Differentiate the equation for node b, and perform a suitable substitution to show that the describing equation for the circuit is

$$\frac{d^2v}{dt^2} + (R+1)\frac{dv}{dt} + (R+4)v = Rv_g + \frac{dv_g}{dt}.$$

(c) Hence, find the natural frequencies of the circuit in the following cases:

(i) $R = 2\Omega$ (ii) $R = 5\Omega$ (iii) $R = 6\Omega$

$(-3/2 \pm j\sqrt{15}/2;\ -3,-3;\ -5,-2)$

Soln:

Differentiate (1)

$$R\frac{dV_g}{dt} + 4\frac{dV_1}{dt} = R\frac{d^2V}{dt} + (R+4)\frac{dV}{dt}$$

Sub in (2)

$$R\frac{dV_g}{dt} + \cancel{4}\frac{dV}{dt} - 4RV_1 = R\frac{d^2V}{dt} + (R+\cancel{4})\frac{dV}{dt}$$

$$\therefore \frac{dV_g}{dt} - 4V_1 = \frac{d^2V}{dt} + \frac{dV}{dt}$$

Substitute (1) in again (sub for $4V_1$)

$$\frac{dV_g}{dt} - R\frac{dV}{dt} - V(R+4) + RV_g = \frac{d^2V}{dt} + \frac{dV}{dt}$$

$$\frac{d^2V}{dt} + (1+R)\frac{dV}{dt} + V(R+4) = RV_g + \frac{dV_g}{dt}$$

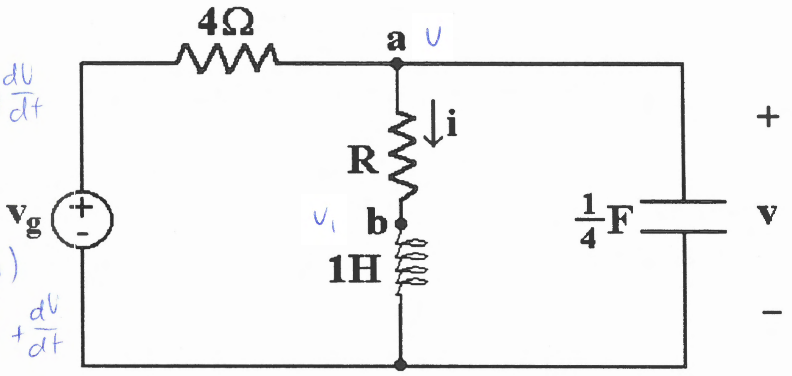


Figure 7.4: Figure for Question 7.3

7.4

(a) Find the forced response if

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = f(t)$$

where f(t) is the forcing function given by

(i) 6 (ii) $8e^{-2t}$ (iii) $6t + 14$

$(2;\ -8e^{-2t};\ 2t + 2)$

(b) If $x(0) = 4$ and $dx(0)/dt = -2$, find the complete solution for x in each of the situations of part (a).

$(2e^{-t} + 2;\ 9e^{-t} - 8e^{-2t} + 3e^{-3t};\ e^{-t} + e^{-3t} + 2t + 2)$

7.5

(a) Find the second order differential equation satisfied by mesh current i_1 in the circuit of Figure 7.5.

$(\frac{d^2i_1}{dt^2} + 7\frac{di_1}{dt} + 6i_1 = 3v_g + 2\frac{dv_g}{dt})$

(b) Let $v_g = 8e^{-2t}$ V, $i_1(0^+) = 2$ A and $i_2(0^+) = 9$ A. Find $\dfrac{di_1(0^+)}{dt}$.

(-19 A/s)

(c) Assuming the values from part (b), find the complete solution for $i_1(t)$.

$(-3e^{-t} + 3e^{-6t} + 2e^{-2t}$ A)

b: $v = L\frac{d}{dt}(i_1 - i_2)$

$v = \frac{d}{dt}i_1 - \frac{d}{dt}i_2$

but $v = V_g - 2i_1$

$\therefore V_g - 2i_1 + \frac{d}{dt}i_2 = \frac{d}{dt}i_1$

Now use Supermesh:

$V_g - 2i_1 - 3i_2 - \frac{d}{dt}i_2 = 0$

Sub into above

$V_g - 2i_1 + V_g - 2i_1 - 3i_2 = \frac{d}{dt}i_1$

$\frac{d}{dt}i_1 = 2V_g - 4i_1 - 3i_2$

$= -19$ A/s

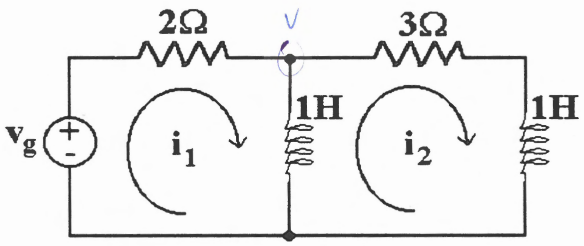


Figure 7.5: Figure for Question 7.5

7.6

Find $x(t)$ for $t > 0$, where

$$\frac{dx}{dt} + 4x + 4\int_0^t x\,dt = f(t) \qquad \text{and} \qquad x(0) = 2$$

and (a) $f(t) = 1$ (b)$f(t) = 2t^2$

$((2 - 3t)e^{-2t};\ 3(1 - t)e^{-2t} + t - 1)$

$\frac{dx(0)}{dt} + 4x(0) + 4\int_0^0 x\,dt = 2(0)^2$

$\frac{dx(0)}{dt} = -8$

7.7

(a) In the circuit of Figure 7.6, the switch has been open for a long time, and is then closed at $t = 0$. Find, at $t = 0^+$:

(i) i (ii) v (iii) di/dt (iv) dv/dt

(0, 0, 2 A/s, 40V/s)

(b) Showing *all* working, for $t > 0$, find: (i) $i(t)$ (ii) $v(t)$.

($i(t) = e^{-2t}(-2\cos(4t) - 0.5\sin(4t)) + 2$ A)

$v(t) = e^{-2t}(-6\cos 4t + 7\sin 4t) + 6$

use KCL

$-2 + C\frac{dv}{dt} + i = 0$

insert the v from KCL law into equation above

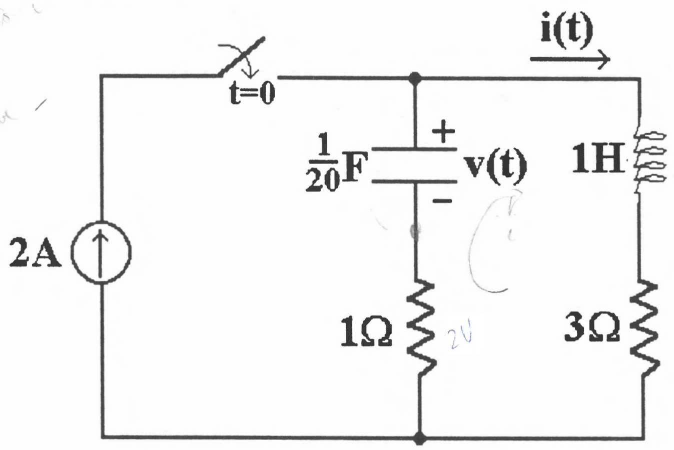


Figure 7.6: Figure for Question 7.7

7.8

Use a computer to produce a printout of the graph of the function

$$v(t) = e^{-2t}(-6\cos(40t) + 7\sin(40t)) + 6 \text{ V}.$$

Print your graph for the range $0 < t < 2$ s. Comment in some detail on the shape of the graph and on how it differs from the answer to Question 7.7(b)(ii). Try to explain how the natural and forced responses are easily seen in the graph, and why the natural response is *transient* in its nature. Can you identify the damping coefficient, see the damped frequency and work out the resonant frequency?

7.9

Find $i(t)$ for $t > 0$ if the circuit of Figure 7.7 is in steady state at $t = 0^-$.
$(200(6e^{-t} - e^{-6t})\text{mA})$

$i(0^-) = 1$

$\frac{di}{dt}(0^-) = 0$

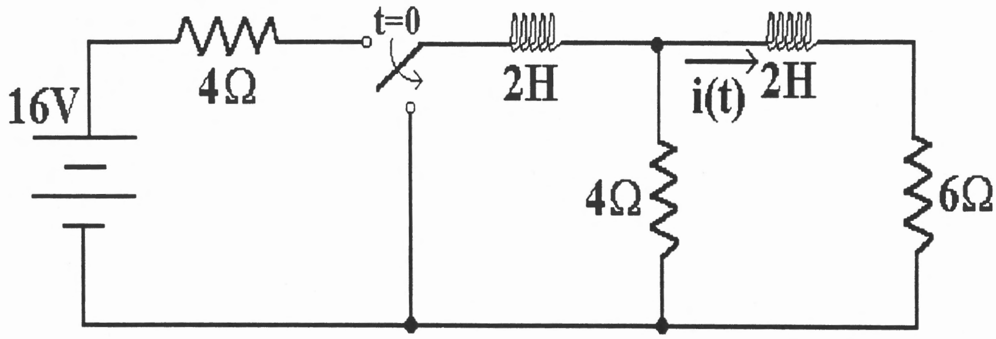


Figure 7.7: Figure for Question 7.9

7.10

Find $v(t)$ for $t > 0$ in the circuit of Figure 7.8, given that there is no initial stored energy in either the capacitor or the inductor.
$((8 - 6t)e^{-2t} - 5\sin(2t) - 8\cos(2t)\text{V})$

minus to plus, unnatural (−)
plus to minus, natural (+)

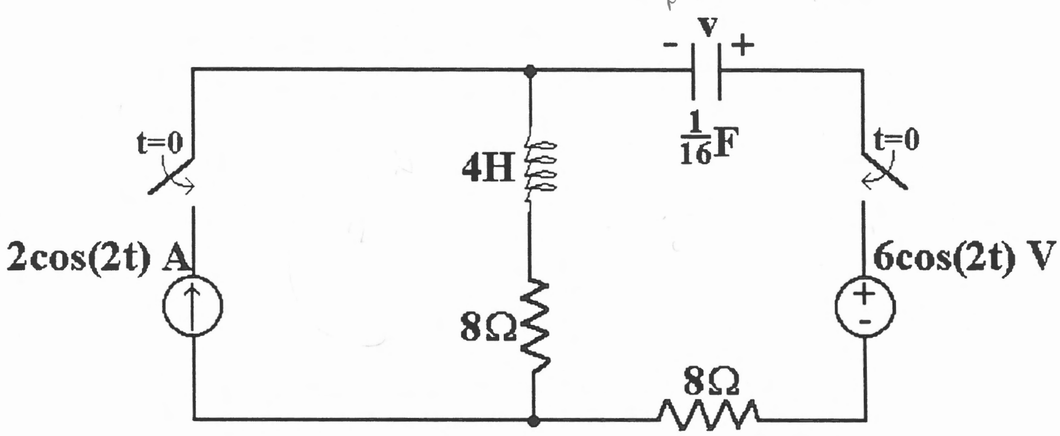


Figure 7.8: Figure for Question 7.10

TUTORIAL 7 (Additional)

7.11

In the circuit of Figure 7.9, R= 1Ω, L= $\frac{4}{3}$H, C= $\frac{1}{4}$F, $V_o = 2$V and $I_o = -3$A. For this circuit:

(a) Write down the integro-differential equation that is obtained from simple nodal analysis of the circuit, and use it to show that

$$\frac{dv(0^+)}{dt} = -\frac{V_o + RI_o}{RC} = 4\ Vs^{-1}$$

(b) Obtain the characteristic equation, and from it show that the voltage is given as a function of time by

$$v(t) = 5e^{-t} - 3e^{-3t}\ \text{V}$$

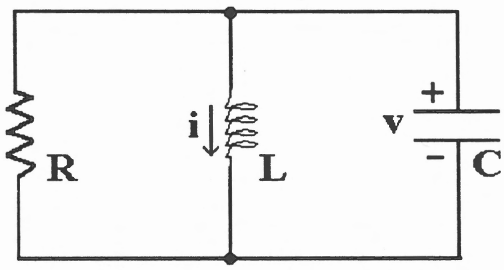


Figure 7.9: Figure for Questions 7.11 to 7.15

7.12

In Figure 7.9, suppose that R= 5Ω, L= 1H, C= $\frac{1}{10}$F, V_o = 0V and $I_o = -\frac{3}{2}$A. Use the same technique as in Question 8.1 to obtain the initial condition $\frac{dv(0^+)}{dt} = 15$ Vs^{-1}, and hence to show that the voltage is given by

$$v(t) = 5e^{-t}\sin(3t).$$

Make a sketch of this function, and write down the values of the damping coefficient and of the damped frequency. Verify that the *resonant frequency* is given by

$$\omega_o = \sqrt{\alpha^2 + \beta^2} = \frac{1}{\sqrt{LC}}.$$

7.13

In a source-free parallel RLC circuit, R= 1kΩ and C= 0.25μF. Find the inductance L, so that the circuit is

(a) overdamped, with $s_1 = -1000s^{-1}$ and $s_2 = -3000s^{-1}$

(b) underdamped, with $\beta = 1000$ rad/s

(c) critically damped.

($\frac{4}{3}$H; $\frac{4}{5}$H; 1H)

7.14

Find the differential equation satisfied by i in Figure 7.9. Use this result to find $i(t)$ for $t > 0$, if R= 10Ω, L= 2H, C= 50mF, $v(0) = 0$ and $i(0) = 6$A.

($\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC}i = 0$; $e^{-t}(6\cos(3t) + 2\sin(3t))$)

7.15

The larger the value of R, the less damping there is in the underdamped case of the parallel RLC circuit (because $\alpha = 1/2RC$). Let $R = \infty$ (i.e. an open circuit) and show that, in the source-free case:

$$\frac{d^2v}{dt^2} + \omega_o^2 v = 0$$

In this case, find the general solution for v(t).

($A_1\cos(\omega_o t) + A_2\sin(\omega_o t)$)

7.16

It is required to find $v(t)$ for $t > 0$ in Figure 7.10, given that $v(0) = 6$V and $i(0) = 2$A. Derive the second order differential equation for the voltage across the capacitor, and hence show that the characteristic equation is

$$s^2 + 2s + 5 = 0$$

From this, write down the general form of the natural response. Next, find the forced response (a simple piece of reasoning will get you there). You already have $v(0) = 6$V as one initial condition, so use the conditions around the capacitor to show that the other initial condition is

$$\frac{dv(0^+)}{dt} = 10 \text{ Vs}^{-1}.$$

Hence, give the complete solution for voltage v as a function of time.
($v(t) = e^{-t}(-4\cos(2t) + 3\sin(2t)) + 10$ V)

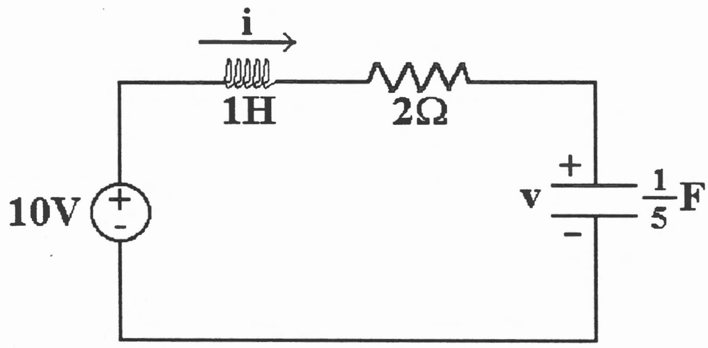


Figure 7.10: Figure for Questions 7.16 and 7.17

7.17

In Figure 7.10, let $v_g = 0$, R= 6Ω, L= 1H, $v(0) = 8$V and $i(0) = 4$A. Find $i(t)$ for $t > 0$ if the capacitance is

(a) $\frac{1}{5}$F

(b) $\frac{1}{34}$F

(c) $\frac{1}{9}$F

($7e^{-5t} - 3e^{-t}$ A; $4e^{-3t}(\cos(5t) - \sin(5t))$ A; $(4 - 20t)e^{-3t}$ A)

7.18

The circuit below is in dc steady state at $t = 0^-$. Find $v(t)$ for $t > 0$.
$(e^{-2000t}(4\cos(4000t) - 3\sin(4000t))$ V)

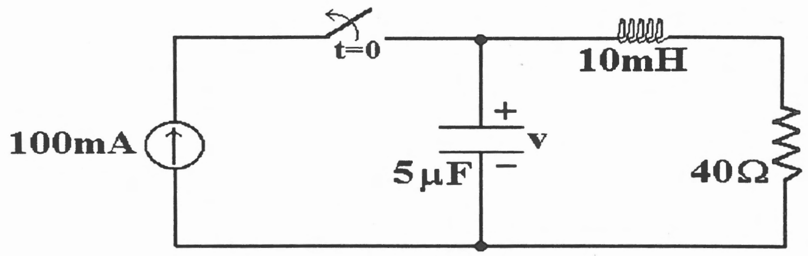


Figure 7.11: Figure for Question 7.18

7.19

Find $v(t)$ for $t > 0$ if the circuit of Figure 7.12 is in dc steady state at $t = 0^-$.
$(8 - e^{-2t}(8\cos(4t) - 6\sin(4t))$ V)

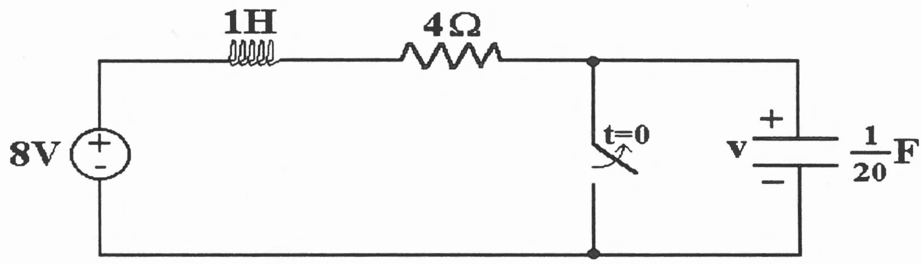


Figure 7.12: Figure for Question 7.19

7.20

Find $i(t)$ for $t > 0$ if $C = \frac{1}{5}F$
$(4 - 5e^{-t} + e^{-5t}$ A)

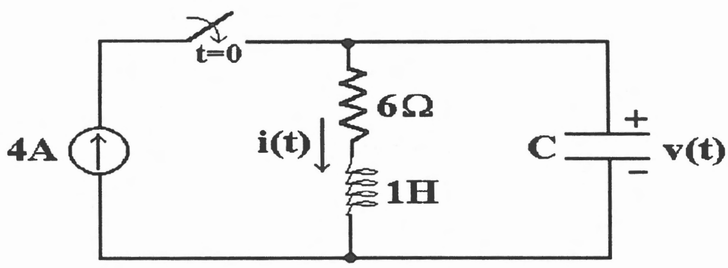
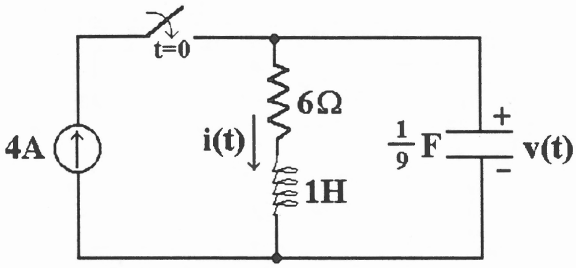


Figure 7.13: Figure for Question 7.20

WORKSHEET 7

7.A

In the circuit below, the switch is closed at $t = 0$ after being open for a long time. Showing all your working carefully, and with clear explanation of each stage, find $i(t)$ for $t > 0$.



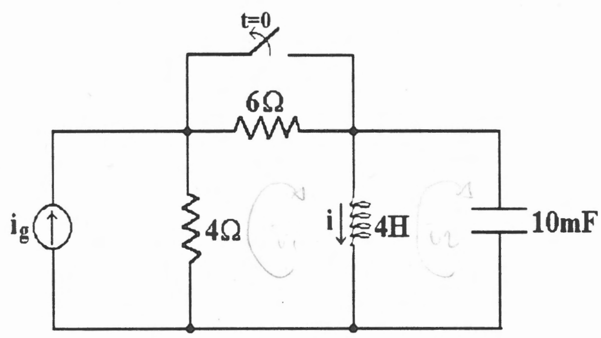
7.B*

In the circuit shown below, find $i(t)$ for $t > 0$, if $i_g = 10$A and the circuit is in dc steady state at $t = 0^-$. You should proceed as follows:

(a) Obtain the differential equation for the circuit, and give it in standard form.

(b) Solve for $i(t)$, showing all of your working.

(c) Distinguish. in your final answer, between the *transient* and the *forced* responses of the current.



7.C

With reference to the circuit shown below:

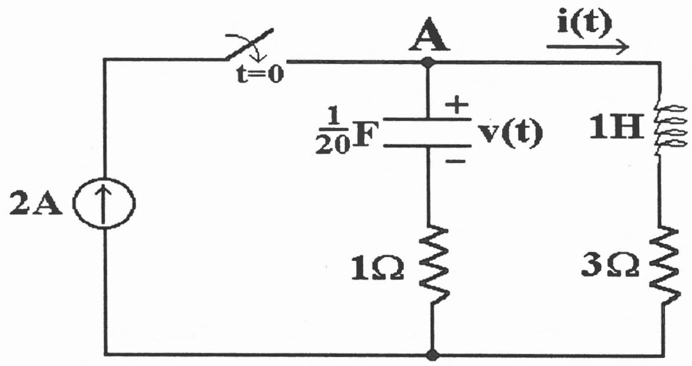
(a) By writing the nodal equation for node A and the mesh equation for the right-hand mesh, show that the differential equation governing $v(t)$ is

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 20v = 120$$

(b) Given that the switch has been open for a long time before $t = 0$, find

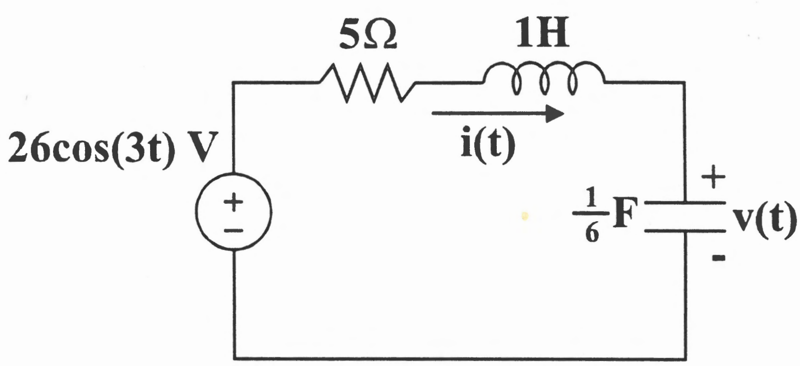
(i) $v(0^+)$ [1] (ii) $\dfrac{dv(0^+)}{dt}$ [2]

(c) Thus, find $v(t)$ for $t > 0$, showing all of your working.



7.D*

In the circuit below, the inductor current is 2A and the capacitor voltage is 6V at $t = 0$. Showing all of your working carefully, and with *clear explanation* of each stage, find $i(t)$ for $t > 0$.



7.E

In the circuit shown below, find $i(t)$ for $t > 0$, given that the circuit is in the dc steady state at $t = 0^-$. You should proceed as follows:

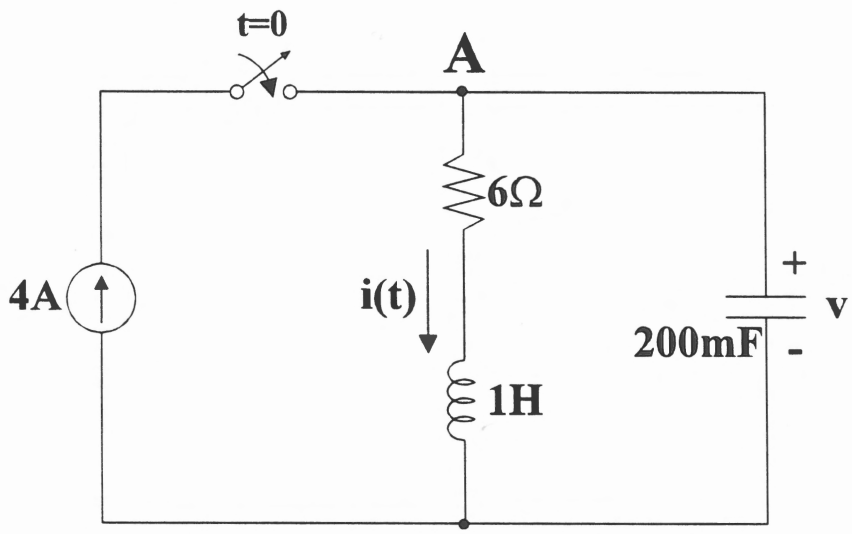
(a) Write the nodal equation for node A and the mesh equation for the right-hand mesh (both for $t = 0^+$), and hence show that the differential equation governing $i(t)$ is

$$\frac{d^2 i}{dt^2} + 6\frac{di}{dt} + 5i = 20$$

(b) Find the initial conditions:

(i) $i(0^+)$ [1] (ii) $\dfrac{di(0^+)}{dt}$

(c) Thus, find $i(t)$ for $t > 0$, showing all of your working.



7.F

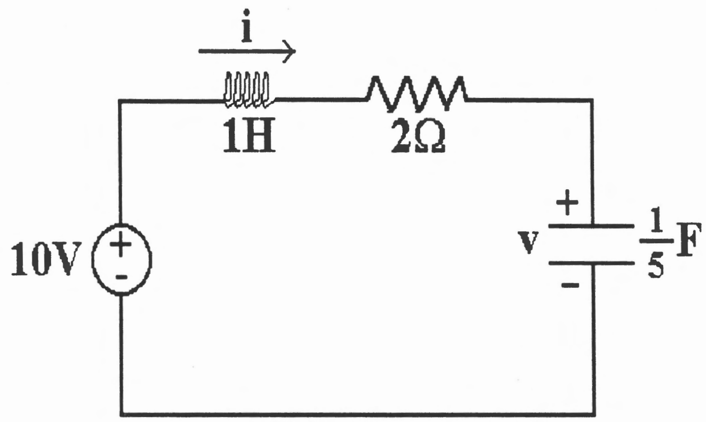
It is required to find $v(t)$ for $t > 0$ in the circuit shown below, given that $v(0) = 6\text{V}$ and $i(0) = 2\text{A}$. Derive the integro-differential equation for the *current* in the circuit from Kirchoff's Voltage Law, and hence show that the characteristic equation is

$$s^2 + 2s + 5 = 0$$

From this, write down the general form of the natural response. Next, find the forced response (a simple piece of reasoning will get you there). You already have $v(0) = 6\text{V}$ as one initial condition, so use the conditions around the capacitor to show that the other initial condition is

$$\frac{dv(0^+)}{dt} = 10\ \text{Vs}^{-1}.$$

Hence, give the complete solution for voltage v as a function of time.



PART 3

Chapter 8

Some Frequency Domain Concepts

In Lectures C16, C17 and C18 we cover:

- Some basic properties of sinusoidal functions and complex numbers
- *Phasors*, and a shorter way of finding the *ac steady state response* of a circuit with a sinusoidal forcing function
- Voltage and current relationships for R, C or L, using phasors
- The *impedance* and *admittance* of resistors, capacitors and inductors
- The idea of the *frequency response* of a circuit, and an introduction to the *frequency domain*
- Circuits which selectively pass either high or low frequency inputs, known as *high-pass filters* and *low-pass filters*
- The meaning of *resonance*, and a frequency-domain treatment of RLC circuits

8.1 Sinusoids and Complex Numbers

Very commonly, the voltage or current generators that drive electric circuits are *sinusoidal*, or *ac*. We therefore need to have some very efficient ways of describing sinusoids to deal with them where they occur. The ultimate motivation for this is that, if a circuit is driven for a long time by an ac

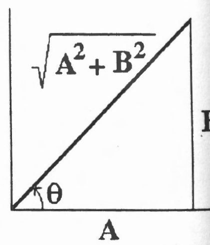
source, then the response of the circuit will eventually settle down to an ac forced response, known as the *ac steady state*. Analysis of any circuit in which alternating current or voltage is applied requires us to be able to find the ac steady state response quickly and easily. This necessitates a few mathematical concepts. We first revise some basic properties of sinusoids, and then go straight on to look at complex numbers.

- A sinusoid can be written $v(t) = V_m \sin(\omega t)$, where V_m is the maximum value that the oscillation attains above the average, and is called the *amplitude*. The *angular frequency*, in radians per second, is ω.

- Sinusoids are *periodic*, since $v(t+T) = v(t)$, and T is called the *period*. We connect period and angular frequency by $\omega = 2\pi/T$. The *frequency* of oscillation is $f = 1/T = \omega/(2\pi)$, measured in *hertz*. Clearly, this also yields the important formula: $\omega = 2\pi f$.

- The sinusoid $v_1(t) = V_m \sin(\omega t + \phi)$ *leads* $v_2(t) = V_m \sin(\omega t)$ by the *phase angle* ϕ. Likewise, $v_2(t)$ *lags* $v_1(t)$ by ϕ, with the phase angle properly in radians but often given in degrees for convenience. To compare phase, v_1 and v_2 must both be sines or both be cosines and both must have positive amplitude.

- The sum of a sine wave and a cosine wave of the same frequency is another sinusoid of the same frequency. Referring to the sketch,we see that



$$A\cos(\omega t)+B\sin(\omega t) = \sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}}\cos(\omega t) + \frac{B}{\sqrt{A^2+B^2}}\sin(\omega t)\right]$$
$$= \sqrt{A^2+B^2}(\cos(\omega t)\cos(\theta) + \sin(\omega t)\sin(\theta))$$
$$= \sqrt{A^2+B^2}\cos(\omega t - \theta),$$

where $\theta = \tan^{-1}\frac{B}{A}$.

- A *complex number* may take the form $A = a + jb$, where $j = \sqrt{-1}$. This is the *rectangular* form of the complex number, and we call a the *real part* and b the *imaginary part* of A, written $a = \Re(A)$ and $b = \Im(A)$.

- A may also be written in the *polar* form $A = |A|e^{j\alpha} = |A|\angle\alpha$. Here $|A|$ is the *magnitude*, given by $|A| = \sqrt{a^2+b^2}$ and α is the *argument*, given by $\alpha = \tan^{-1}\frac{b}{a}$.

cos(θ) = s (90+θ)
sin(θ) = cos(90-θ)

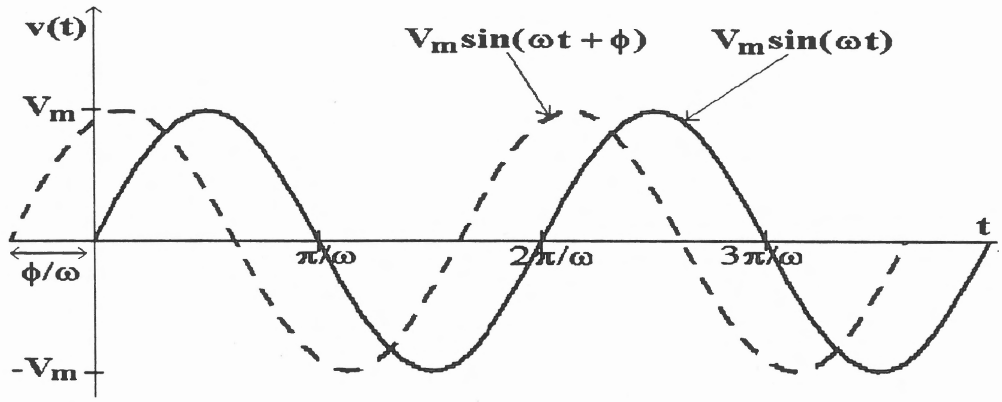


Figure 8.1: Sinusoidal Functions and their Phase Relationships

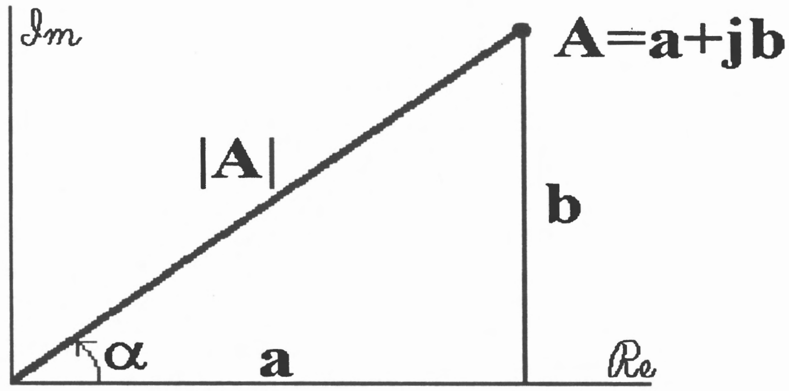


Figure 8.2: Geometrical Representation of the Complex Number A

- The two forms are summarised in the *Argand diagram* of Figure 8.2

8.2 Complex Excitations and Phasors

Your work with finding the complete responses of first and second order circuits should have convinced you that it is easier to work with exponential forcing functions than with sinusoidal ones. This is simply because with sinusoidal functions it was necessary to deal with the sine and cosine terms separately, whereas exponential functions have derivatives which are just multiples of themselves. Complex numbers offer us a way to treat sinusoids as though they were exponentials, thereby greatly simplifying calculations

must make sure function is cos ~~sign~~ before we convert to polar.

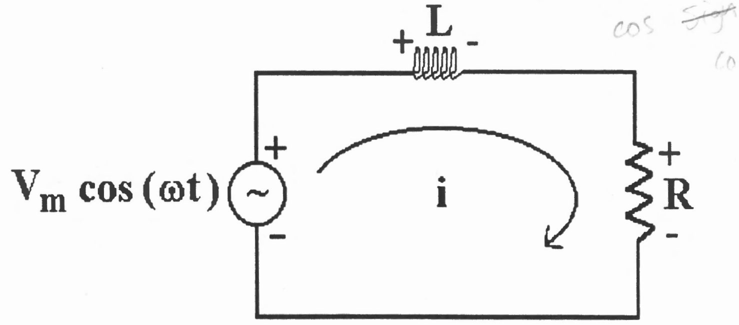


Figure 8.3: A Simple RL Circuit, Driven by a Sinusoidal Voltage Source

in ac-driven circuits and opening up some very powerful ways of analysing such circuits, as we shall now see.

Suppose we are dealing with a sinusoidal voltage $v = V_m \cos(\omega t + \theta)$. The complex number $\mathbf{V} = V_m e^{j\theta} = V_m \angle\theta$ has exactly the same amplitude and phase as v, and so we can use it to represent the sinusoid. By Euler's Formula, $\mathbf{V} = V_m \cos(\theta) + jV_m \sin(\theta)$, and $\mathbf{V}e^{j\omega t} = V_m \cos(\omega t + \theta) + jV_m \sin(\omega t + \theta)$. Hence, our sinusoid can be given by $v = \Re(\mathbf{V}e^{j\omega t})$. When we use a complex number to carry amplitude and phase information in this way, we call it a *phasor*, and print it in boldface type.

You will need to become fluent at converting sinusoids to phasors, and vice versa. Note that the frequency of the driving source is assumed known, so that these conversions are very easy. At 8 rad/s, for instance $\mathbf{V} = 5\angle 30^o$V represents the sinusoid $v(t) = 5\cos(8t + 30^o)V$, while the sinusoidal current $i(t) = 5\sin(2t + 20^o)$ mA is re-written as $5\cos(70^o - 2t) = 5\cos(2t - 70^o)$ and is then represented by $\mathbf{I} = 5\angle(-70^o)$, with $\omega = 2$ rad/s implied. In general, the sinusoid $v(t) = V_m \cos(\omega t + \phi)$ is represented by $v = \mathbf{V}e^{j\omega t}$, where $\mathbf{V} = V_m \angle\phi$, and in this way the frequency information is also conveyed.

The use of phasors involves a movement into the world of complex numbers as far as the mathematics is concerned, but you must remember that the physical voltages and currents which interest us as engineers are all *real*. Whenever you use phasors, therefore, be sure to take the *real part* of them when relating their meaning back to the circuit you are analysing.
We now aim to show how phasors can simplify the analysis of a circuit. Consider the first-order circuit in Figure 8.3. The describing equation is

seen to be $L\frac{di}{dt} + Ri = V_m \cos(\omega t)$, from which the natural response is $i_n(t) = A_1 e^{-Rt/L}$.

If we now apply our familiar techniques to finding the forced response, things become a little complicated. The trial solution is chosen from the table to be $i_f(t) = A\cos(\omega t) + B\sin(\omega t)$, which we can differentiate and substitute into the describing equation to get:

$$L(-\omega A \sin(\omega t) + \omega B \cos(\omega t) + R(A\cos(\omega t) + B\sin(\omega t)) = V_m \cos(\omega t)$$

Equating terms in sines and in cosines gives the simultaneous equations

$$RA + \omega LB = V_m$$

and

$$-\omega LA + RB = 0$$

from which

$$A = \frac{RV_m}{R^2 + \omega^2 L^2}$$

and

$$B = \frac{\omega L V_m}{R^2 + \omega^2 L^2}$$

The forced response is therefore

$$i_f(t) = \frac{RV_m}{R^2 + \omega^2 L^2}\cos(\omega t) + \frac{\omega L V_m}{R^2 + \omega^2 L^2}\sin(\omega t),$$

and we could write it as a single sinusoidal function using the rules for sinusoids discussed earlier:

$$i_f(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}\cos\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

Clearly, the forced response is a sinusoid (as predicted by the trial solution), but equally clearly this method of obtaining it is now getting unwieldy! Contrast the new method of solution, which uses phasors:

The describing equation is $L\dfrac{di}{dt} + Ri = V_m\cos(\omega t) = \mathbf{V}e^{j\omega t}$, and we know it has a sinusoidal solution, which may be written $i(t) = \mathbf{I}e^{j\omega t}$. Substituting this into the defining equation now gives:

$$j\omega L\mathbf{I}e^{j\omega t} + R\mathbf{I}e^{j\omega t} = \mathbf{V}e^{j\omega t}$$

so

$$j\omega L\mathbf{I} + R\mathbf{I} = \mathbf{V}$$

Hence $$\mathbf{I} = \frac{\mathbf{V}}{R + j\omega L} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \Big\langle -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

and so $$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{j(\omega t - tan^{-1}(\omega L/R))}$$

The actual current is the real part of this expression, which is easily seen:

$$i_f(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right),$$

and this is exactly the forced solution that we had before.

8.3 Voltage and Current Phasor Relations

Phasors greatly simplify the treatment of resistors, capacitors and inductors in ac circuit analysis by allowing them all to be treated as though Ohm's Law applied. Consider an element with voltage $v = V_m \cos(\omega t + \theta)$ across it and through which current $i = I_m \cos(\omega t + \phi)$ flows.

- If the element is a resistor, then $v = Ri$. If we apply the *complex* voltage $V_m e^{j(\omega t+\theta)}$, the complex current which results is $I_m e^{j(\omega t+\phi)}$, so $V_m e^{j(\omega t+\theta)} = RI_m e^{j(\omega t+\phi)}$. Dividing through by $e^{j\omega t}$ now gives us $V_m e^{j\theta} = RI_m e^{j\phi}$, which simply reduces to

  $$\underline{\mathbf{V} = R\mathbf{I}}$$

  Note that since R is just a scalar number, the above equations give $\theta = \phi$, which means that the phases of the voltage and the current are the same, so we say for a resistor that voltage and current are *in phase*.

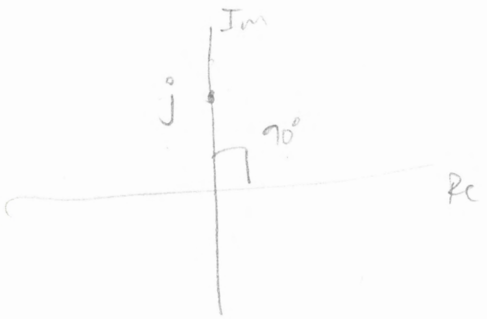
- If the element is a capacitor, then $i = C\frac{dv}{dt}$. Substituting in our complex voltage and current as above, we get

  $$\begin{aligned} I_m e^{j(\omega t+\phi)} &= C\frac{d}{dt}\left[V_m e^{j(\omega t+\theta)}\right] \\ &= j\omega C V_m e^{j(\omega t+\theta)} \end{aligned}$$

  Dividing through by $e^{j\omega t}$ and converting to phasors now gives

  $$\underline{\mathbf{V} = \frac{\mathbf{I}}{j\omega C}}$$

when you divide by (j), you shift a complex number by 90°



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in cap, I leads V
V leads I in inductor

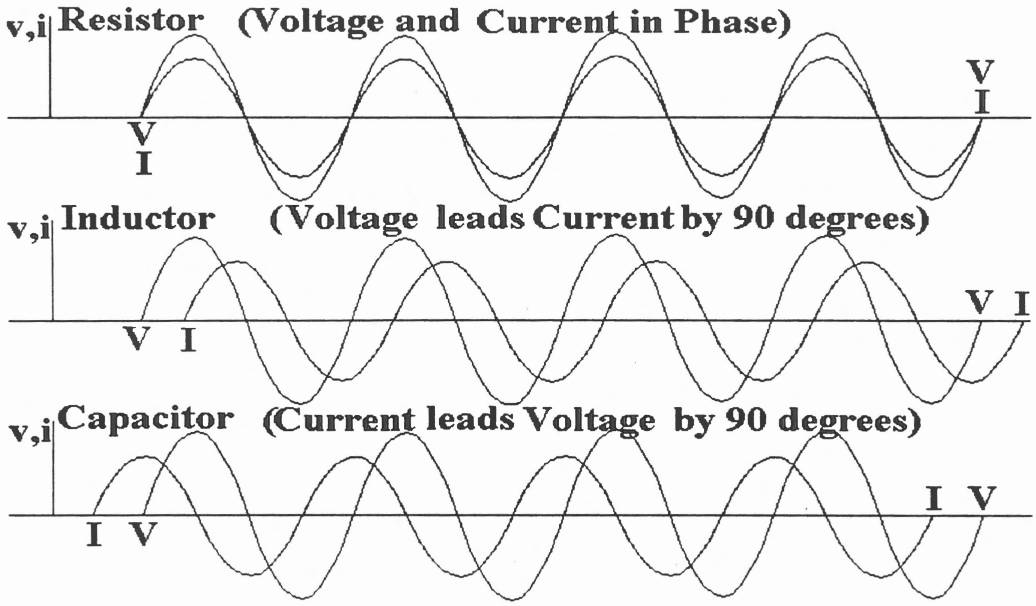


Figure 8.4: Phase Relationships in Resistors, Capacitors and Inductors

- If the element is an inductor, then $v = L\frac{di}{dt}$. Once again we can apply complex voltage and current, and we get

$$\begin{aligned} V_m e^{j(\omega t+\theta)} &= L\frac{d}{dt}\left[I_m e^{j(\omega t+\phi)}\right] \\ &= j\omega L I_m e^{j(\omega t+\phi)} \end{aligned}$$

Dividing as usual by $e^{j\omega t}$ and re-writing the expression in phasors:

$$\underline{\mathbf{V} = j\omega L\mathbf{I}}$$

Note that, since $j = 1\angle 90^o$, the voltage across a capacitor has a phase of 90^o less than the current through it (i.e. $\theta = \phi - 90^o$). Conversely, in an inductor the voltage has phase of 90^o greater than the current. Hence:

In a capacitor the current leads the voltage by 90^o, and
in an inductor the voltage leads the current by 90^o

These phase relationships are illustrated in Figure 8.4.

We now *define* the ratio of the phasor voltage to the phasor current for an element or group of elements as the *impedance* Z of the element(s), where

In general, $\vec{U} = \vec{I} Z$ (like ohms law)

for resistor, $Z = R + j0$ (Z is impedance purely real)

capacitor, $Z = 0 + \frac{1}{j\omega C}$ (Z imaginary + negative) $= \frac{-j}{\omega C}$ injects phase of −90 degrees on sig signal input across it

inductor, $Z = 0 + j\omega L$ (Z imaginary)

$\vec{I} = \vec{U}(Y)$

$Y = \frac{1}{Z}$ is called the adamittance

$Z = R + jX$ complex number; R: Resistance; X: Reactance

$Y = G + jB$; G: Conductance; B: susceptance

$$Z = \frac{\mathbf{V}}{\mathbf{I}}$$

The unit of impedance is evidently the ohm (Ω) since $\mathbf{V}$ is in volts and $\mathbf{I}$ is in amps. Z is a complex number, but it is *not* a phasor, because it does not carry amplitude and phase information which represents some time-varying sinusoid. However, as a complex number it can be written in polar form:

$$Z = |Z|\angle\theta_z = \frac{V_m}{I_m}\angle(\theta - \phi)$$

The impedance therefore has a magnitude and an argument. It can also be written in rectangular form:

$$Z = R + jX$$

where $R = \Re(Z)$ is known as the *resistive component* or *resistance*, and $X = \Im(Z)$ is the *reactive component* or *reactance*. You can easily see that the following relations apply:

$$|Z| = \sqrt{R^2 + X^2} \qquad \theta_z = \tan^{-1}\frac{X}{R} \qquad R = |Z|\cos(\theta_z) \qquad X = |Z|\sin(\theta_z)$$

We can now go back to our analysis of the resistor, capacitor and inductor voltage-current relations, and find the impedance of each component:

- $Z_R = R$, so the impedance of a resistor is purely resistive, with $X_R = 0$
- $Z_C = \frac{1}{j\omega C} = -\frac{j}{\omega C}$, so the impedance of a capacitor is purely reactive, with $R_C = 0$ and $X_C = -\frac{1}{\omega C}$. Capacitive reactance is always a negative number, because ω and C are positive by definition
- $Z_L = j\omega L$, so the impedance of an inductor is also purely reactive, with $R_L = 0$ and $X_L = \omega L$. Inductive reactance is a positive quantity.

The inverse of the impedance Z is called the *admittance* Y of an element or group of elements, measured in siemens (S) or mhos (℧). Thus $Y = 1/Z$. Admittance is also a complex number, and its rectangular form is $Y = G + jB$, where G is the *conductance* of the element, and B is its *susceptance*. Resistors have real admittance (conductance) only, capacitors have positive susceptance only and inductors have negative susceptance only.

8.4 The Low-Pass Filter

We now introduce a means of looking at the performance of a circuit which is completely different from any of the circuit analysis techniques that we

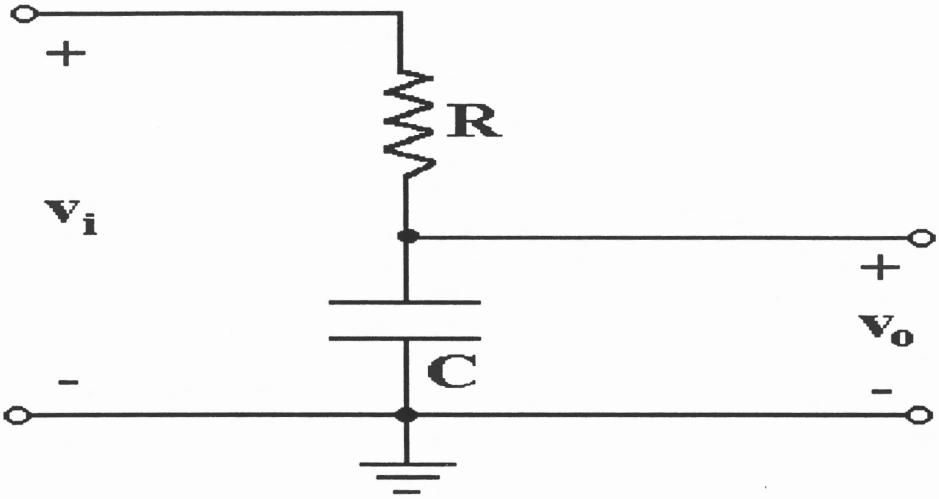


Figure 8.5: A Simple Low-Pass Filter

have met in the course so far. Up until now, we have analysed many circuits which have inputs that are time-varying voltages or currents, and we have found output voltages or currents within these circuits that have also been *functions of time.* In certain cases, we have even sketched the outputs on graphs whose horizontal axis represented the advance of time.

Engineers often find it very useful to supplement or to replace this kind of *time-domain analysis* with what is known as the *frequency-domain analysis* of a circuit. In brief, this involves calculating how an output of the circuit will respond if the *frequency* of the input (or inputs) is varied. Graphs which illustrate this sort of analysis give the frequency of the input along the horizontal axis, while quantities of interest, such as the magnitude of the output, are then plotted against the vertical axis.

As a first example, let us think about applying a sinusoidal signal of frequency ω to the input terminal of the circuit of Figure 8.5. It should now be evident to you that we can regard the two impedances (the resistor and the capacitor) as forming a voltage divider for the ac input signal, and so we may write

$$H(j\omega) = \frac{\mathbf{V}_\mathrm{o}}{\mathbf{V}_\mathrm{i}} = \frac{1/(j\omega C)}{1/(j\omega C) + R} = \frac{1}{1 + j\omega RC}$$

Transfer function

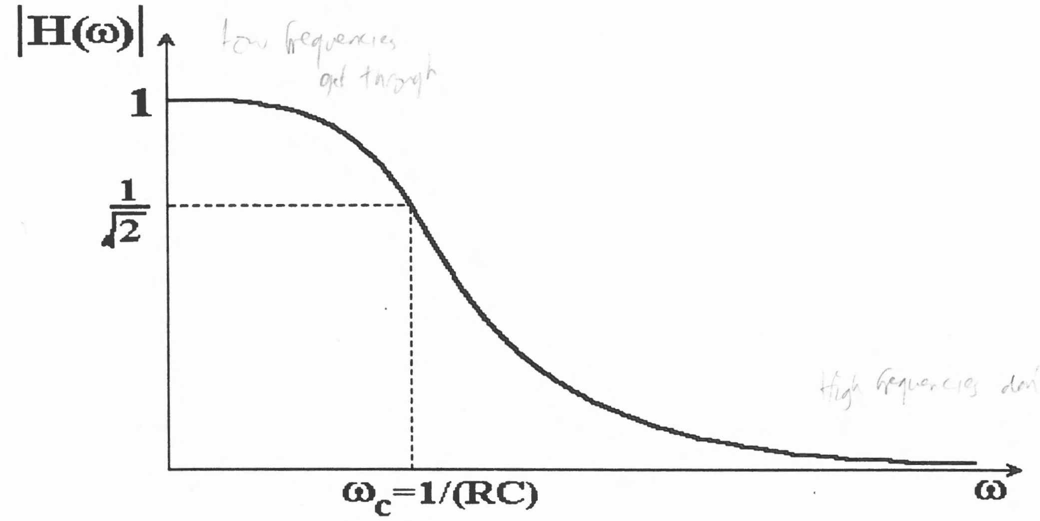


Figure 8.6: A Low-Pass Filter Amplitude Response

We call this relationship between the output and the input of a circuit its *transfer function*. You will see that the transfer function is a function of the frequency, ω, of the input to the circuit, and also of j, and so we write

$$\mathbf{H}(j\omega) = \frac{1}{1 + j\omega RC} \text{ so } |\mathbf{H}(j\omega)| = \left|\frac{\mathbf{V_o}}{\mathbf{V_i}}\right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

complex number

Now, notice in this case that

- as $\omega \to 0$, $|\mathbf{H}(j\omega)| \to 1$. i.e. $|\mathbf{V_o}| \to |\mathbf{V_i}|$
- as $\omega \to \infty$, $|\mathbf{H}(j\omega)| \to 0$. i.e. $|\mathbf{V_o}| << |\mathbf{V_i}|$

If we sketch the magnitude of the transfer function, with ω on the horizontal axis, we get the graph shown in Figure 8.6, which is often known as the *amplitude response* of the circuit. It is clear from this graph that the output voltage is only a substantial fraction of the input voltage if the input is a sinusoid of low frequency. Any high-frequency input to this circuit will *not* appear (or will be greatly reduced) at the output, and we say that the circuit has *filtered* it out. Since this particular circuit only passes low-frequency inputs through to the output, filtering out the higher frequencies, we call the circuit a *low-pass filter*. Perhaps you can see that this makes sense in another way: at low frequencies, the capacitor in Figure 8.5 acts

$\omega_c \rightarrow$ when half the power of input appears at output

Power is related to V^2

like an open circuit, and so the output voltage is very similar to the input voltage; at high frequencies, the capacitor acts like a short circuit, so the output voltage is at ground potential, regardless of the amplitude of the high-frequency input.

It is usual to regard the highest frequency that the filter passes as being that frequency for which the power of the output ($\propto v_o^2$) is reduced to *half* the power of the input ($\propto v_i^2$). In other words, the so-called *cut-off frequency*, ω_c, is found from

$$|\mathbf{H}(j\omega_c)|^2 = \left|\frac{\mathbf{V_o}}{\mathbf{V_i}}\right|^2 = \left|\frac{1}{1+j\omega_c RC}\right|^2 = \frac{1}{|1+j\omega_c RC|^2} = \frac{1}{(1+\omega_c^2 R^2 C^2)} = \frac{1}{2}$$

and from this we see that $2 = 1 + \omega_c^2 R^2 C^2$, and so

$$\omega_c = \frac{1}{RC}$$

Thus, for instance, if the circuit above had $R = 1\text{k}\Omega$ and $C = 1\mu\text{F}$, then the cut-off frequency for the circuit (above which it effectively filters out inputs) would be

$$\omega_c = \frac{1}{1000 \times 10^{-6}} = 1000 \text{ rads}^{-1}$$

which equates to a frequency in hertz of

$$f_c = \frac{\omega_c}{2\pi} = 159.15 \text{ Hz}$$

8.5 High-Pass and Other Filters

Now consider swapping the positions of the resistor and the capacitor in the circuit of Figure 8.5, and instead taking the output as shown in Figure 8.7. Once again, thinking in terms of voltage division, for an input frequency ω we can write

$$\mathbf{H}(j\omega) = \frac{\mathbf{V_o}}{\mathbf{V_i}} = \frac{R}{1/j\omega C\ + R} = \frac{j\omega RC}{1+j\omega RC}$$

In this case, we observe that

- as $\omega \to 0$, $|\mathbf{H}(j\omega)| \to 0$. i.e. $|\mathbf{V_o}| << |\mathbf{V_i}|$
- as $\omega \to \infty$, $|\mathbf{H}(j\omega)| \to 1$. i.e. $|\mathbf{V_o}| \to |\mathbf{V_i}|$

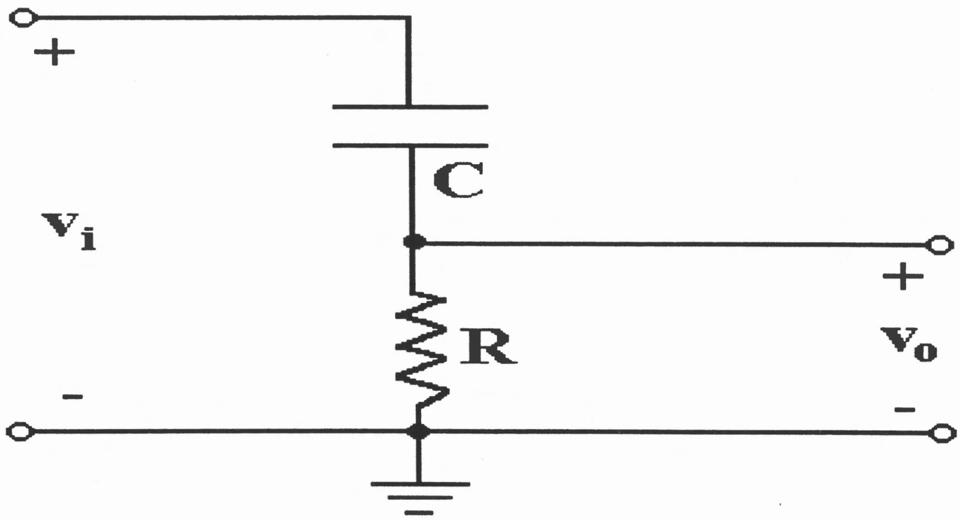


Figure 8.7: A High-Pass Filter

This time a sketch of the frequency response looks like the one shown in Figure 8.8, where the output voltage is only a substantial fraction of the input voltage if the input is at a sufficiently *high* frequency. Low-frequency inputs are filtered out by the circuit and only the higher frequencies are passed. The circuit is, therefore, known as a *high-pass filter*. Once again, this ought to make sense if you think about how the capacitor performs: at high frequencies it behaves as a short circuit, effectively tying the output voltage to the input voltage; at low frequencies, it behaves as an open circuit, and since no current then flows in the resistor, the output is at ground potential, regardless of the amplitude of the input.

As in the low-pass case, the cut-off frequency is easy to find. It occurs where

$$|\mathbf{H}(j\omega_c)|^2 = \left|\frac{\mathbf{V_o}}{\mathbf{V_i}}\right|^2 = \left|\frac{j\omega_c RC}{1 + j\omega_c RC}\right|^2 = \frac{\omega_c^2 R^2 C^2}{1 + \omega_c^2 R^2 C^2} = \frac{1}{2}$$

Hence, $2\omega_c^2 R^2 C^2 = 1 + \omega_c^2 R^2 C^2$, and so

$$\omega_c = \frac{1}{RC}$$

just as in the low-pass case.

At this point it is worth noting that if a high-pass filter (HPF) and a low-pass filter (LPF) are *cascaded* (i.e. if they are connected in series as shown

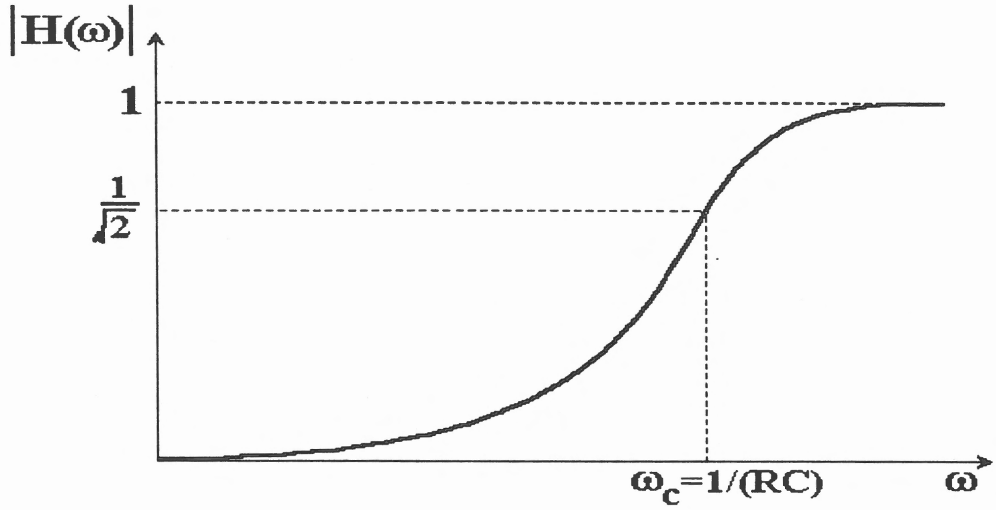


Figure 8.8: A High-Pass Filter Amplitude Response

in Figure 8.9), then, depending upon the relative values of the cut-off frequencies of the high-pass and low-pass sections, certain other useful filters can result.

If the cut-off frequency of the HPF is lower than that of the LPF ($\omega_{ch} < \omega_{cl}$) then the result is a *band-pass filter*, whose amplitude response is shown in Figure 8.10(a). Such a filter has the useful property that it will only pass a certain range of input frequencies (the ones that lie between ω_{cl} and ω_{ch}), and it will reject all other frequencies. The range of frequencies that the filter passes is called the *bandwidth* of the band-pass filter, and is written

$$B = \omega_{ch} - \omega_{cl}$$

By contrast, consider the case where $\omega_{ch} > \omega_{cl}$ (see Figure 8.10(b)). A filter with this kind of amplitude response is known as a *band-stop filter*, and you can see that it has the ability to reject frequencies that lie between ω_{cl} and ω_{ch} while passing all other frequencies. The rejected frequency range is known as the *stop-band*. In later courses you will study how to design more sophisticated versions of such filters, which play an extremely useful rôle in many circuits by "notching out" unwanted frequencies, such as 50Hz mains "hum".

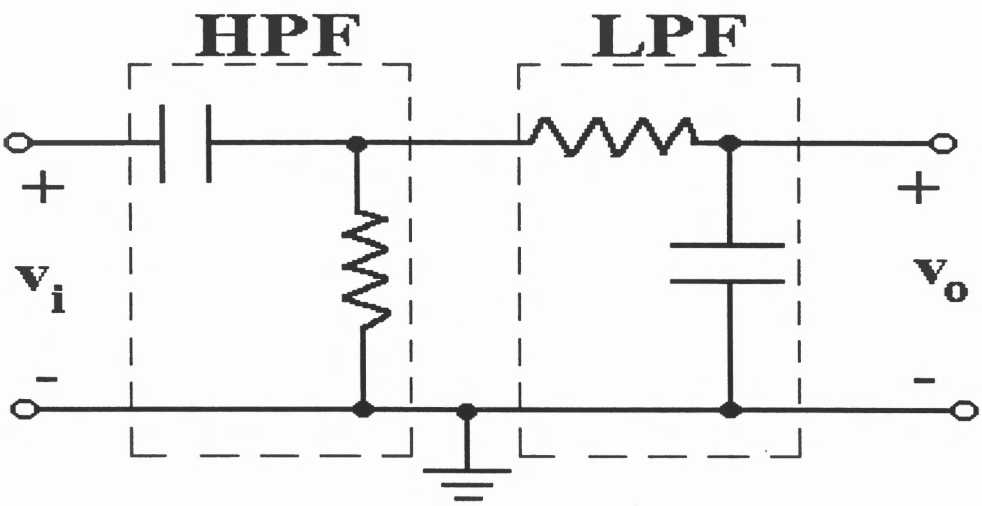


Figure 8.9: Cascaded High-Pass and Low-Pass Filters

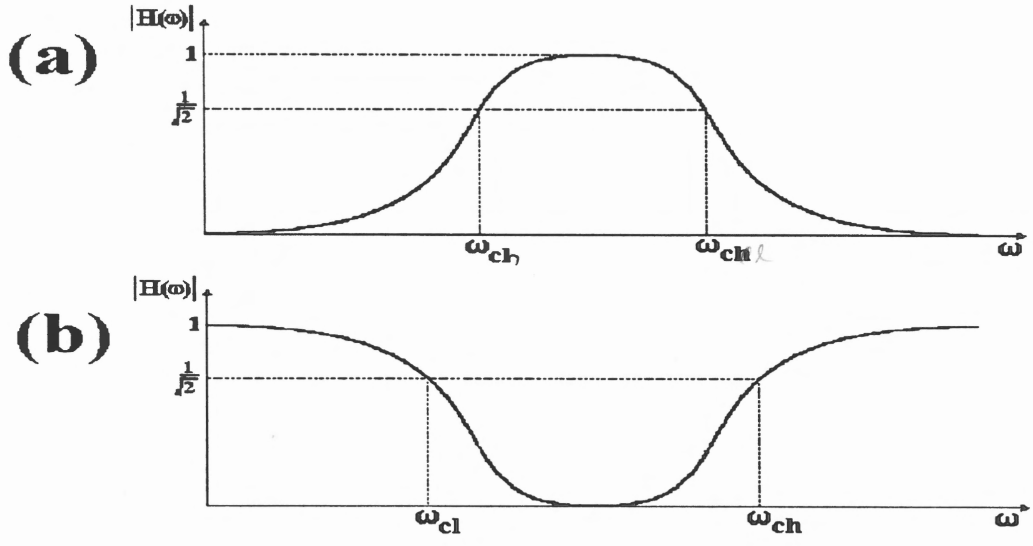


Figure 8.10: (a) A Band-Pass Filter (b) A Band-Stop Filter

Notch Filters.

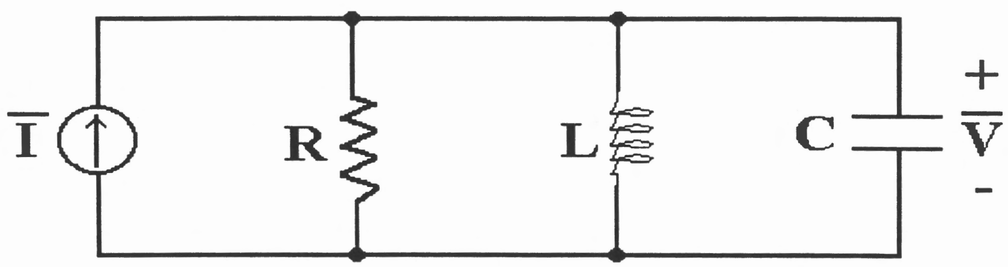


Figure 8.11: An RLC Parallel Circuit

8.6 Resonance

We have already analysed the RLC parallel circuit in the *time-domain*, and you should recall that we identified how such circuits will respond to an input by *oscillating*, if their component values are related by the inequality $L < 4R^2C$ (that is, if the circuit is *underdamped*). Let us now revisit the RLC parallel circuit (see Figure 8.11) and analyse its performance in the *frequency-domain*. Let the input to the circuit be denoted by the phasor **I**, and let the circuit's output be taken as the sinusoidal voltage across the capacitor, written as the phasor **V**. The transfer function is then the ratio of output to input, thus:

parallel

$$\mathbf{H}(j\omega) = \frac{\mathbf{V}}{\mathbf{I}} = Z_{tot} = R \parallel j\omega L \parallel 1/j\omega C = \frac{1}{1/R \; + \; 1/j\omega L \; + \; j\omega C}$$

$Z_R \parallel Z_L \parallel Z_C$

Separating the real and imaginary parts in the denominator, it becomes clear that

$$\mathbf{H}(j\omega) = \frac{1}{(1/R) + j[\omega C - (1/\omega L)]}$$

and hence that the *magnitude* of the frequency response (the amplitude response) is

$$|\mathbf{H}(j\omega)| = \frac{1}{\sqrt{(1/R)^2 + (\omega C \; - \; 1/\omega L)^2}}$$

From this, you can see that the output ($|\mathbf{V}|$) will be greatest in relation to the input ($|\mathbf{I}|$) when

$$\omega C - \frac{1}{\omega L} = 0$$

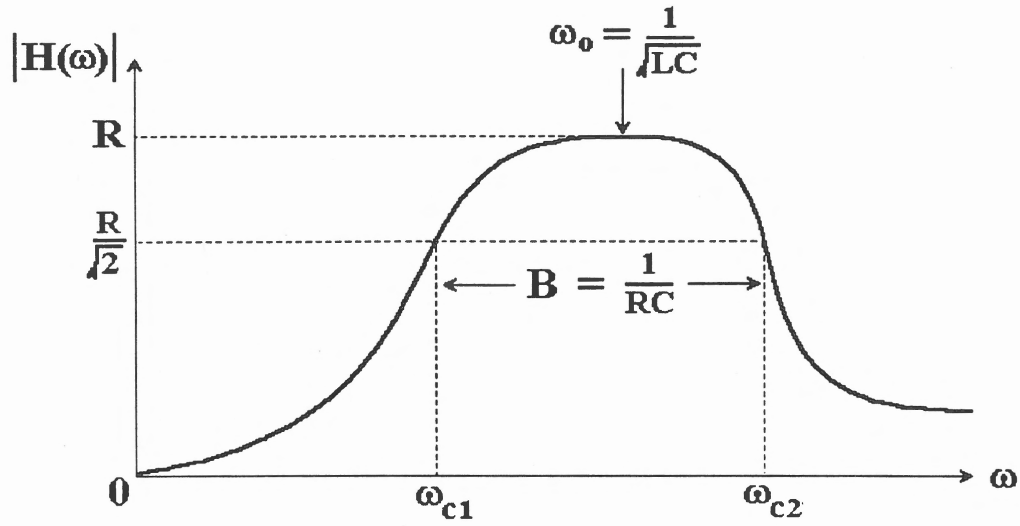


Figure 8.12: Amplitude Response of a Parallel RLC Circuit

This occurs if $\omega C = 1/\omega L$, that is when

$$\omega = \frac{1}{\sqrt{LC}}$$

We have already seen that this is ω_o, the *resonant frequency* of the circuit. Notice that if the input frequency, ω, is equal to ω_o, then $|\mathbf{H}(j\omega)| = R$: in other words, at resonance the only impedance offered by the circuit is due to its *resistive* element, since the capacitive and inductive impedances cancel.

A plot of $|\mathbf{H}(j\omega)|$ is given in Figure 8.12. You should confirm from the above equation for $|\mathbf{H}(j\omega)|$ that

- when $\omega = 0$, $|\mathbf{H}(j\omega)| = 0$
- as $\omega \to \infty$, $|\mathbf{H}(j\omega)| \to 0$

The bandwidth of the amplitude response can also be seen from the graph, and it is evident that the parallel RLC circuit acts as a band-pass filter, responding strongly to frequencies within the bandwidth and around ω_o, the resonant frequency, but rejecting both very low and very high frequency inputs. Now, recalling that the definition of the bandwidth is that part of the frequency range between the cut-off frequencies where

$$|\mathbf{H}(j\omega)| = \frac{|\mathbf{H}(j\omega)_{max}|}{\sqrt{2}}$$

we can calculate the cutoff frequencies by solving for ω the equation

$$|\mathbf{H}(j\omega)| = \frac{1}{\sqrt{(1/R)^2 + (\omega C - 1/\omega L)^2}} = \frac{R}{\sqrt{2}}$$

This equation gives us

$$(\omega C - \frac{1}{\omega L})^2 = \frac{1}{R^2}$$

and hence

$$\omega C - \frac{1}{\omega L} = \pm\frac{1}{R}$$

which can be written in the form

$$C\omega^2 \pm \frac{1}{R}\omega - \frac{1}{L} = 0$$

Now, this equation has four roots (two for each sign in the coefficient of ω), but, since we are not interested in negative frequencies, we shall consider only the two positive roots - these are the two cut-off frequencies of the filter. Hence, applying the quadratic formula to the previous equation with a positive coefficient of ω gives us

$$\omega_{c1} = \frac{-\frac{1}{R} + \sqrt{\frac{1}{R^2} + \frac{4C}{L}}}{2C}$$

Likewise, applying the quadratic formula to the previous equation with a negative coefficient of ω, and again considering only the positive root, we get

$$\omega_{c2} = \frac{\frac{1}{R} + \sqrt{\frac{1}{R^2} + \frac{4C}{L}}}{2C}$$

Hence, finally, we can obtain the bandwidth of the RLC parallel circuit. It is, very simply,

$$B = \omega_{c2} - \omega_{c1} = \frac{1}{RC}$$

Some RLC circuits resonate only in a very narrow peak around ω_o, while others have outputs that will oscillate in response to quite a wide range of input frequencies around ω_o. The sharpness of the resonant peak in the plot

of $|\mathbf{H}(j\omega)|$ is called the *Quality factor*, or Q, of the circuit, and it can be calculated from

$$Q = \frac{\omega_o}{B}$$

As a rule of thumb, a circuit with a Q-factor above 5 is generally regarded as a highly-selective filter, while circuits with Q-factors below 5 have very broad peaks in the frequency domain. Although you will meet up with Q again in later courses, you should already be able to show that, for the parallel RLC circuit, since $\omega_o = 1/\sqrt{LC}$ and since $B = 1/RC$,

$$Q = \frac{\omega_o}{B} = \omega_o RC = R\sqrt{\frac{C}{L}} = \frac{R}{\omega_o L}$$

A similar development is possible for the series RLC circuit, and you are strongly advised to try the mathematics for yourself. Once you can generate the same argument for the series RLC circuit as the one presented here for the parallel RLC case, you will have gained a good grasp of the frequency domain and you will be well on your way towards using frequency-domain analysis to further your understanding of electrical circuits.

Major Hint: Do the whole lot for series RLC circuit.

TUTORIAL 8

8.1

Consider the following sinusoidal quantities:

(I) $v(t) = 4\sin(3t + 12^o)$ V

(II) $i(t) = 7\cos(4t) + 24\sin(4t)$ A

(III) $p(t) = (4\sqrt{3} - 3)\cos(2\pi t + 30^o) + (3\sqrt{3} - 4)\cos(2\pi t + 60^o)$ mW

(a) In each case, find

(i) the amplitude
(ii) the cosinusoidal phase
(iii) the angular frequency
(iv) the frequency in Hz
(v) the period

and *use this information* to draw a clear, labelled sketch of the function.
(4V, -78^o, 3 rad/s, 0.477Hz, 2.09s;
25A, -73.7^o, 4 rad/s, 0.637Hz, 1.57s;
5mW, 36.9^o, 6.28 rad/s, 1Hz, 1s))

(b) Calculate whether sinusoid (I) leads or lags the function $5\cos(3t - 28^o)$, and by how many degrees.
(I lags by 50^o)

(c) Calculate whether sinusoid (II) leads or lags the function $3\sin(4t + 3^o)$, and by how many degrees.
(II leads by 13.3^o)

8.2

(a) Find the phasor representation of:

(i) $4\cos(2t + 45^o)$
(ii) $8\cos(2t) + 15\sin(2t)$
(iii) $-2\sin(5t - 65^o)$

(b) Find the time-domain functions represented by the phasors:

(i) $10\angle - 17^o$
(ii) $6 + j8$
(iii) $-j6$

(In all cases, ω = 3rad/s)

($4\angle 45^o$; $17\angle - 61.9^o$; $2\angle 25^o$)
($10\cos(3t - 17^o)$; $10\cos(3t + 53.1^o)$; $6\cos(3t - 90^o)$)

8.3

Using phasors, find the time-domain representation of the ac steady state current i, if $v = 12\cos(1000t + 30^o)$ V and the shaded block represents

(a) A 4kΩ resistor
(b) A 15mH inductor
(c) A 500nF capacitor

($3\cos(1000t + 30^o)$ mA; $800\cos(1000t - 60^o)$ mA; $6\cos(1000t + 120^o)$ mA)

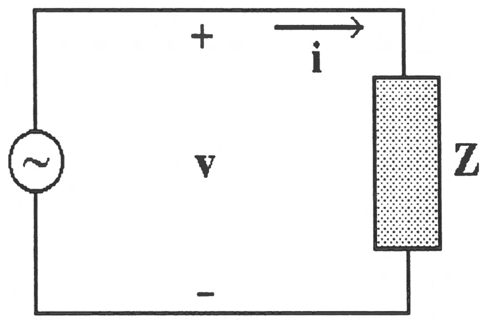


Figure 8.13: Figure for Questions 8.3 and 8.4

8.4

An ac current $i(t) = 3\sin(5\pi \times 10^6 t - 20^o)$ μA flows in the circuit of Figure 8.13. Use phasors to find, in time-domain form, the voltage $v(t)$ across the shaded element, if the shaded block represents

(a) A $1\text{M}\Omega$ resistor

(b) A 4mH inductor

(c) A 27pF capacitor

Note: Leave π in the frequency of answers but **not** *in the amplitude*
($3\sin(5\pi.10^6 t - 20^o)$V; $188\sin(5\pi.10^6 t + 70^o)$mV; $7.07\sin(5\pi.10^6 t - 110^o)$mV)

8.5

In Figure 8.14, use phasors, impedance and voltage division to find the forced ac response of the capacitor voltage if the input voltage amplitude, A, is:

(a) 34 V (b) 17 V half input

($10\cos(4t - 28.1^o)$ V; $5\cos(4t - 28.1^o)$ V)

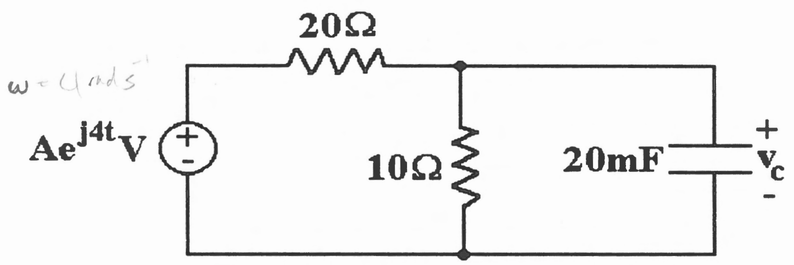


Figure 8.14: Figure for Question 8.5

8.6

(a) Show by means of a circuit diagram how you would construct a low-pass filter from a $100\text{k}\Omega$ resistor and a 10nF capacitor. If the input and output of the filter are regarded as voltages, calculate both $\mathbf{H}(j\omega)$ and $|\mathbf{H}(j\omega)|$, showing your working carefully. State the cut-off frequency of the filter, and then draw the graph of its transfer function, showing the correct position for the cut-off frequency.

(b) Show that $\mathbf{H}(j\omega) = -2\omega^2/(8 - \omega^2 + 4j\omega)$ is the transfer function of a high-pass filter, and find $|\mathbf{H}(j\omega)_{max}|$ and ω_c for this filter. Show both of these quantities on a neat sketch of the transfer function.
($\frac{1000}{1000+j\omega}$, $\frac{1}{\sqrt{1+\omega^2/10^6}}$, $\omega_c = 159$Hz; 2, 2.83 rad/s)

8.7
Find the ac steady state value of i if:

(a) $\omega = 1$ rad/s (b) $\omega = 2$ rad/s (c) $\omega = 4$ rad/s

In which of the three cases is the circuit resonating, and why?

($2\cos(t + 36.9^o)$ A; $2.5\cos(2t)$ A; $2\cos(4t - 36.9)$ A)
(The response in (b) has highest amplitude and is in phase with the source)

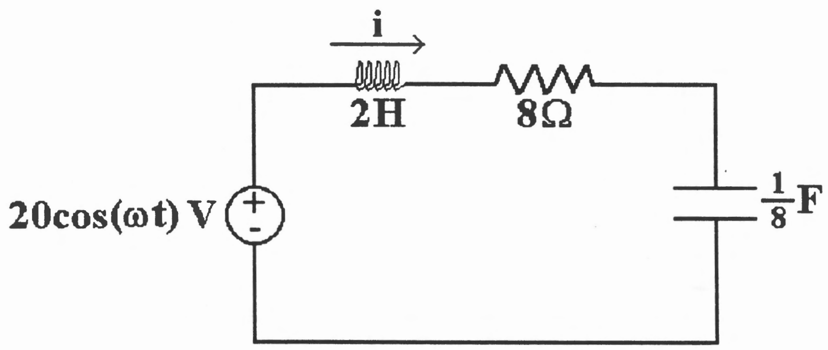


Figure 8.15: Figure for Question 8.7

8.8
For the circuit shown, find Z_{eq} and use the result to find phasor current **I**. If $\omega = 7$ rad/s, find the forced response $i(t)$ corresponding to **I**.

($4\angle 53.1^o$ Ω; $2\angle - 53.1^o$ A; $2\cos(7t - 53.1^o)$ A)

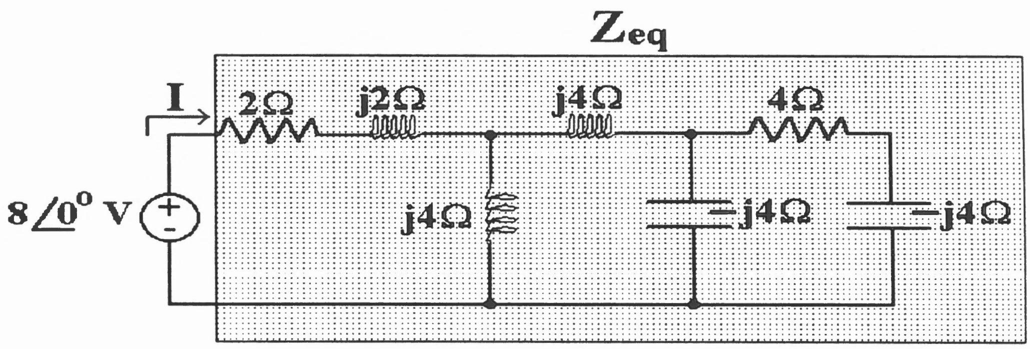


Figure 8.16: Figure for Question 8.8

8.9

Find the reactance X necessary to ensure that the impedance seen by the source in Figure 8.17 is real. If this is the case, find the ac steady-state current $i(t)$ corresponding to **I** if $\omega = 10$ rad/s.

($X = -7.2\ \Omega$; $i(t) = 5\cos(10t)$ A)

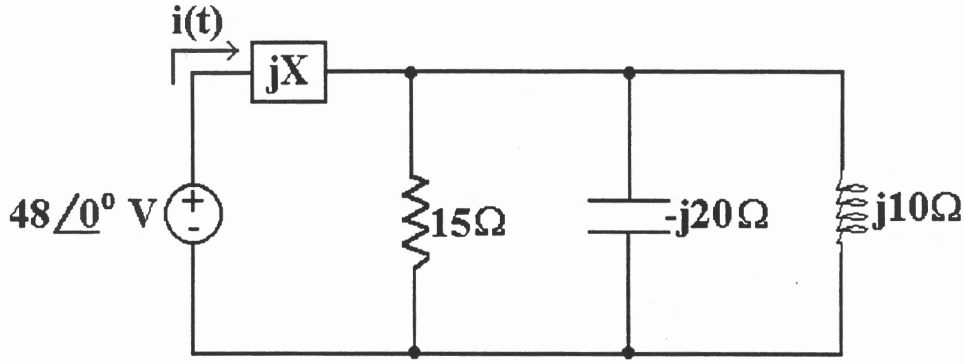


Figure 8.17: Figure for Question 8.9

8.10

Use frequency-domain analysis to find $\mathbf{H}(j\omega)$ and $|\mathbf{H}(j\omega)|$ for the series RLC circuit shown. You should regard **I** as the circuit's output and **V** as its input. From $|\mathbf{H}(j\omega)|$, find an expression for the resonant frequency of the circuit, ω_o, and also find $|\mathbf{H}(j\omega)|_{max}$. Now calculate the values of the two cut-off frequencies of the circuit, and hence state the bandwidth. Finally, show that your calculations lead to two expressions for Q: $Q = \omega_o L/R$ and $Q = 1/(\omega_o RC)$.

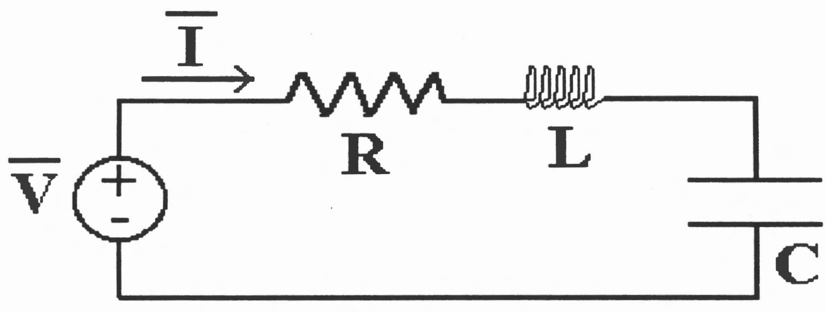
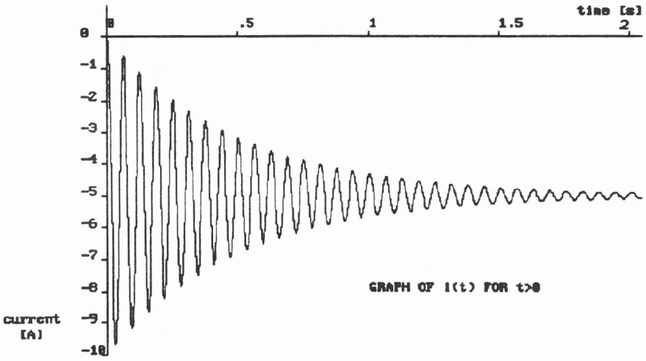


Figure 8.18: Figure for Question 8.10

WORKSHEET 8

8.A*

The graph shown here is the complete response of a current to an event in



a circuit. State, or calculate (giving units where you can):

(a) The time constant, τ

(b) The damping coefficient, α

(c) The damped frequency, β

(d) The forced response, $i_f(t)$ [2]

(e) The resonant frequency, ω_o

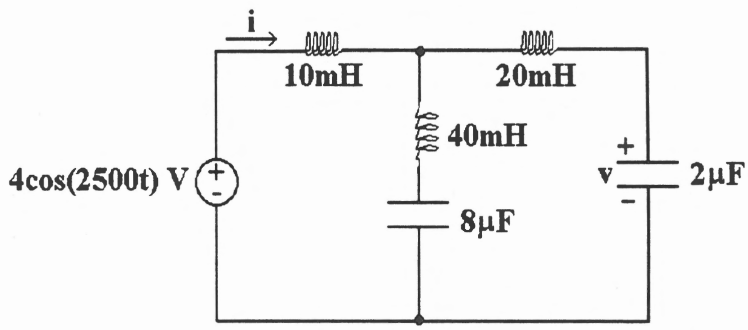
(f) The natural frequencies, $s_{1,2}$

The equation of this function takes the form $\underline{i(t) = e^{k_1 t}(k_2 \cos(k_3 t)) + k_4 \text{ A}}$. State the values of k_1, k_2, k_3 and k_4.

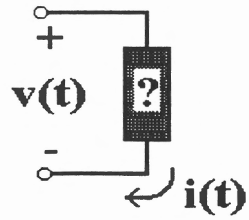
NB Answers (a) to (d) are "round numbers". Note for part (c) $\omega = 2\pi f$

8.B

Draw the phasor equivalent circuit to the circuit shown below. Hence, find the ac steady state values of $i(t)$ and $v(t)$, by calculating the equivalent impedance seen by the source, and applying current division.



8.C*



The element shown has voltage $v(t)$ across it and current $i(t)$ through it, where:

$v(t) = \sin(t)$ and $i(t) = -2\cos(t)$

(a) Find and simplify an expression for the instantaneous power $p(t)$ absorbed by the element.

(b) Show that the instantaneous power oscillates with twice the frequency of the current or the voltage.

(c) What is the average power absorbed by the element?

(d) State the rms values of voltage and current.

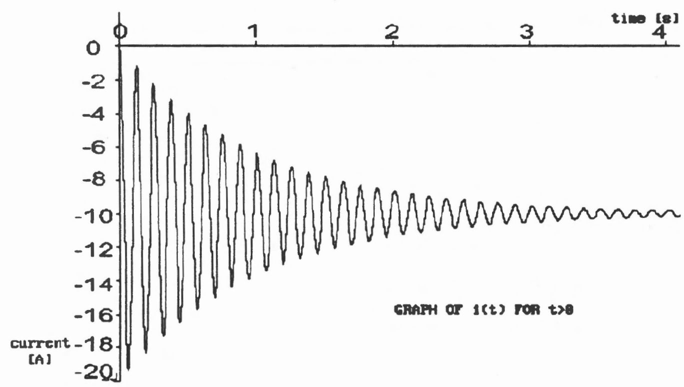
(e) Make a neat sketch, on a single pair of axes, of $v(t)$ and $i(t)$.

(f) Use your sketch to state what kind of component (resistor, capacitor or inductor) the given element is.

(g) Now state the value of the given element, with units.

8.D

The graph above is the complete response of a current to an event in a



circuit. State, or calculate (giving units where you can):

(a) The time constant, τ

(b) The damping coefficient, α

(c) The damped frequency, β

(d) The forced response, $i_f(t)$

(e) The resonant frequency, ω_o

(f) Natural frequencies, $s_{1,2}$

The equation of this function takes the form $\underline{i(t) = e^{k_1 t}(k_2\cos(k_3 t)) + k_4 \text{ A}}$. State the values of k_1, k_2, k_3 and k_4.

NB Answers (a) to (d) are "round numbers". For part (c) $\omega = 2\pi f$

Chapter 9

Phasors in AC Steady-State Analysis

In Lectures C20 and C21 we cover:

- AC steady state circuit analysis using phasors
- Kirchhoff's Laws and rules for impedance combination in AC circuits
- The analysis of circuits with complex sources and impedances, which is called *phasor circuit analysis*
- AC Nodal and Mesh Analysis
- AC Network Theorems, reintroducing the concepts of *superposition*, *Thèvenin's theorem* and the *Principle of Proportionality*

9.1 Kirchhoff's Laws & Complex Impedances

Kirchhoff's Laws hold for phasors as well as for their corresponding time-domain voltages or currents. We can see this by observing that if a complex excitation, say $V_m e^{j(\omega t+\theta)}$, is applied to a circuit, then complex voltages, such as $V_1 e^{j(\omega t+\theta_1)}$, $V_2 e^{j(\omega t+\theta_2)}$ etc., appear across the elements in the circuit. Since Kirchhoff's laws hold in the time domain, KVL applied around a typical loop results in an equation such as

$$V_1 e^{j(\omega t+\theta_1)} + V_2 e^{j(\omega t+\theta_2)} + \ldots + V_N e^{j(\omega t+\theta_N)} = 0$$

Dividing out the common factor $e^{j\omega t}$, we have

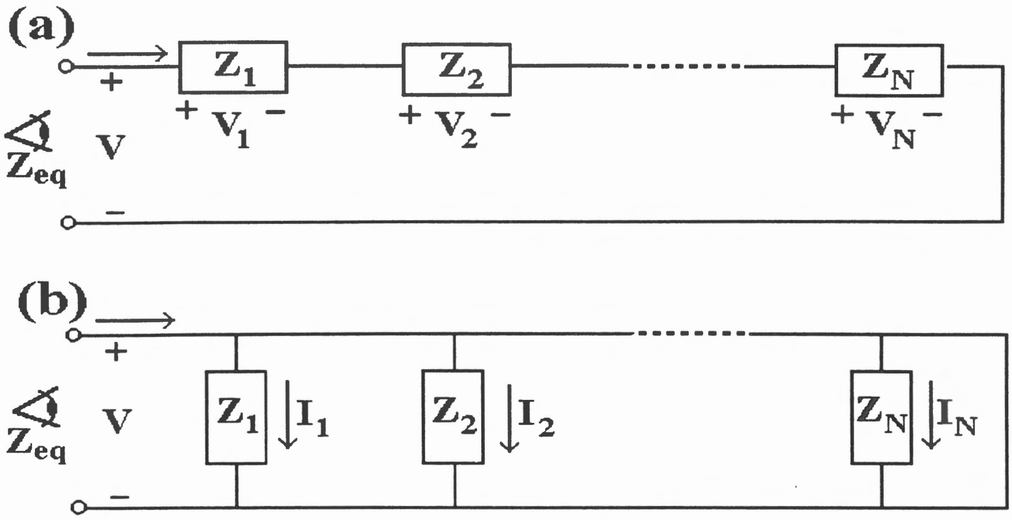


Figure 9.1: (a) Impedances in Series (b) Impedances in Parallel

$$\mathbf{V_1} + \mathbf{V_2} + \ldots + \mathbf{V_N} = 0$$

where $\mathbf{V_n} = V_n \angle \theta_n$, for $n = 1, 2, \ldots, N$ are the phasor voltages around the loop. Thus KVL holds for phasors. A similar development will also establish KCL for phasors.

If we now wish to analyse a circuit that is being excited at a particular frequency ω, and we are interested in the *forced* (or ac steady state) response, we can build up a set of analysis rules, starting with Kirchhoff's laws, just as we did early in the course for dc circuit analysis. We will find that *ac steady-state analysis is identical to resistive circuit analysis, with impedances replacing resistances and phasors replacing time-domain quantities.* As a first example, in the circuit of Figure 9.1(a), phasor current $\mathbf{I}$ flows in each of the impedances, which are connected in series. The voltages across the elements are $\mathbf{V_1} = Z_1\mathbf{I}$, $\mathbf{V_2} = Z_2\mathbf{I}$ up to $\mathbf{V_N} = Z_N\mathbf{I}$. Now applying KVL gives

$$\begin{aligned}
\mathbf{V} &= \mathbf{V_1} + \mathbf{V_2} + \ldots + \mathbf{V_N} \\
&= (Z_1 + Z_2 + \ldots + Z_N)\mathbf{I} \\
&= Z_{eq}\mathbf{I}
\end{aligned}$$

We can therefore see that, for impedances in series

$$Z_{eq} = Z_1 + Z_2 + \ldots + Z_N$$

Similar analysis to this will show that the equivalent impedance for parallel impedances follows the pattern we would expect from our study of parallel resistances, and, in particular, for two impedances in parallel

$$Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2}.$$

Note that it is sometimes more convenient to deal with *admittances* rather than with impedances. The equivalent admittance of several admittances in series is given by

$$\frac{1}{Y_{eq}} = \frac{1}{Y_1} + \frac{1}{Y_2} + \ldots + \frac{1}{Y_N}$$

while the equivalent admittance of several admittances in parallel is found from

$$Y_{eq} = Y_1 + Y_2 + \ldots + Y_N.$$

You will probably now be realising that voltage and current division rules exist in phasor circuits that hold in exactly the same way as they do for simple resistive combinations. Thus, for example, in Figure 9.1(a)

$$\mathbf{V_1} = \frac{Z_1}{Z_1 + Z_2 + \ldots + Z_N}\mathbf{V} = \frac{Z_1}{Z_{eq}}\mathbf{V}$$

while in Figure 9.1(b) with $N = 2$

$$\mathbf{I_1} = \frac{Z_2}{Z_1 + Z_2}\mathbf{I}.$$

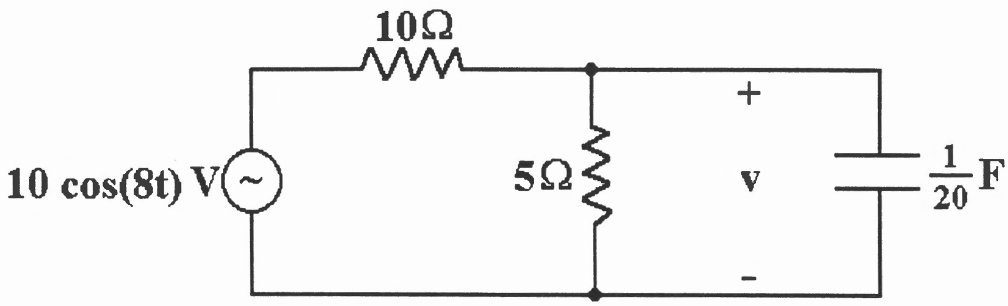
You should be able to derive these and similar relationships from the algebra of simple phasor circuits.

9.2 Analysis of Phasor Circuits

We can now apply all of the above rules to the analysis of circuits with sinusoidal inputs and consisting of any types of impedance. We do this by replacing the circuit with its *phasor equivalent*, and then applying phasor analysis techniques. The process will be illustrated in the examples which follow.

Example 9.1

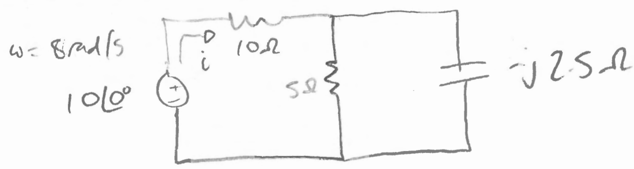
Use phasors to find the current drawn by this circuit, and hence the forced response $v(t)$. Confirm your result by using voltage division.



Step 1: Convert the source to a phasor and the element values to impedances.

$\vec{V} = 10\angle 0^\circ \qquad Z_C = \frac{1}{j\omega C} = -j2.5\,\Omega \qquad \omega = 8$

Step 2: Draw the *phasor circuit*



Step 3: Find Z_{eq} for this circuit

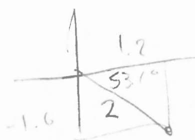
$Z_{eq} = 11 - j2\ \Omega$

$Z_{eq} = 5\Omega \,//\, -j2.5 + 10\Omega$

$\frac{-j12.5}{5-j2.5} = \frac{-j5}{2-j} = \frac{-(2+j)j5}{5} = \frac{-10j + 5}{5} = 1 - j2$

Step 4: Obtain the phasor current $\mathbf{I}$

$\vec{I} = \frac{\vec{V}_{in}}{Z} = \frac{10+j0}{11-j2}\cdot\frac{11+j2}{11+j2} = \frac{110+j20}{125}$ A



Step 5: Find $\mathbf{V}$ and hence $v(t)$

$\vec{V} = \vec{V}_{in} - 10\vec{I}$

$= 10 - \frac{110-j20}{125}\times 10 = \frac{10-1100+j200}{125} = 1.2 - j1.6\,\text{V} = 2\angle -53.1^\circ$

Finally: Find the impedance of the RC parallel pair, and hence check $v(t)$ (by voltage division)

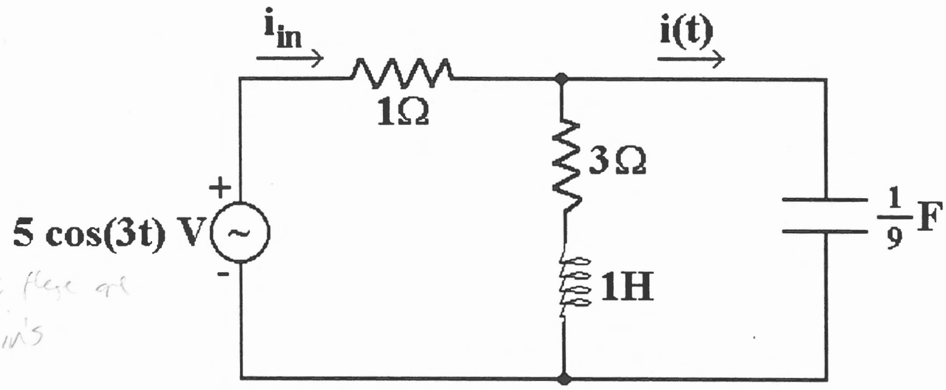
$v(t) = 2\cos(8t - 53.1^\circ)$ ✓

$\vec{V} = \frac{\overset{1-j2}{Z_{RC}}}{Z_{RC}+10}\cdot V_{in}$ by voltage division

$= \frac{1-j2}{11-j2}\cdot\frac{11+j2}{11+j2}\cdot 10 = \frac{11+j2-11j2+2}{125}\cdot 10 = \frac{13-20j}{125}\cdot 10$

Example 9.2

Use phasor methods to find the forced response of the capacitor current $i(t)$.



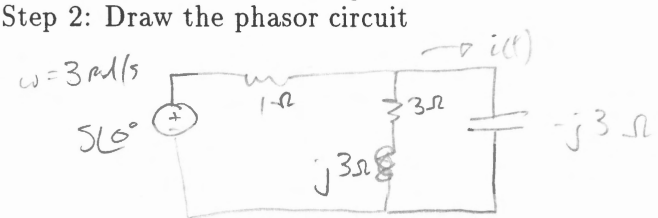
watch out if phase are sin's

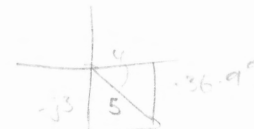
Step 1: Convert the source and elements

$\vec{V} = 5\angle 0^\circ$ $\quad Z_L = j\omega L = j3$ $\quad Z_C = \frac{1}{j\omega C} = -j3$

$\omega = 3$

Step 2: Draw the phasor circuit





Step 3: Find Z_{eq} for the circuit

$Z' = (j3+3) \parallel -j3 = \frac{(3+j3)(-j3)}{j3+3-j3} = \frac{-j9+9}{3} \Omega = 3-j3\ \Omega$

$Z_{eq} = 1 + Z' = 4 - j3\ \Omega$

$Z_{eq} = 5\angle -36.9^\circ$ not a phasor, it's a complex no.

Step 4: Find the phasor current $\mathbf{I_{in}}$

$\vec{I}_{in} = \frac{\vec{V}}{Z_{eq}} = \frac{5\angle 0^\circ}{5\angle -36.9^\circ} = 1\angle 36.9^\circ = 0.8 + j0.6$

Step 5: Use current division to obtain $\mathbf{I}$, and hence $i(t)$.

$\vec{I} = \frac{3+j3}{3+j3-j3} \cdot 0.8 + j0.6 = (1+j)(0.8+j0.6)$

$= 0.8 + 0.6j + 0.8j - 0.6$

$= 0.2 + 1.4j$

$i(t) = 1.41 \cos(3t + 82^\circ)$ A

Look out for shift

9.3 AC Nodal and Mesh Analysis

We have seen that Ohm's Law, KVL, KCL, voltage and current division are all valid analytical tools in phasor circuits. It should therefore be no surprise that further techniques, such as nodal and mesh analysis also have corresponding phasor methods which are valuable for predicting the forced behaviour of ac circuits. These methods follow very smoothly from the basic techniques that we studied in conventional resistive circuits, with the only differences being that all elements are now referred to by their *impedances* instead of just by resistances, and that the unknown quantities for which we solve are now going to be phasors, rather than simple real voltages or currents.

To illustrate the nodal and mesh methods, we will analyse the same circuit, using each technique in turn, and then we will confirm that the two results agree. The circuit to be analysed is shown in Figure 9.2.

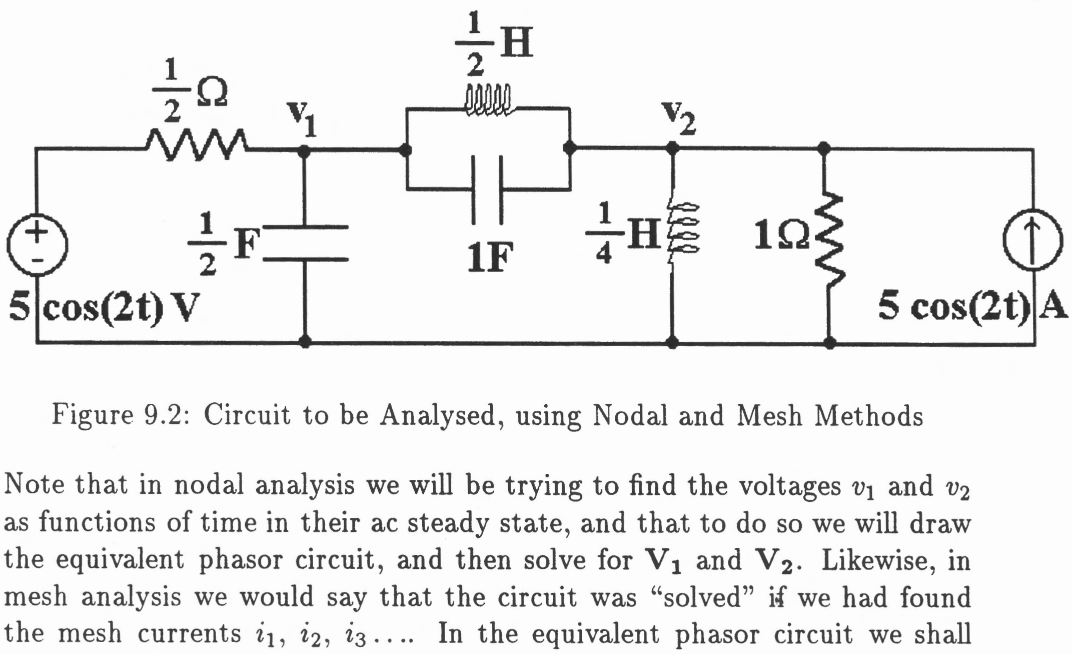


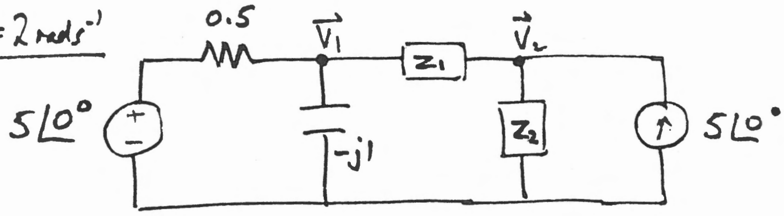
Figure 9.2: Circuit to be Analysed, using Nodal and Mesh Methods

Note that in nodal analysis we will be trying to find the voltages v_1 and v_2 as functions of time in their ac steady state, and that to do so we will draw the equivalent phasor circuit, and then solve for $\mathbf{V_1}$ and $\mathbf{V_2}$. Likewise, in mesh analysis we would say that the circuit was "solved" if we had found the mesh currents i_1, i_2, $i_3 \ldots$. In the equivalent phasor circuit we shall therefore find $\mathbf{I_1}$, $\mathbf{I_2}$, $\mathbf{I_3} \ldots$, and then obtain the time-domain steady state response of the various currents from these phasor quantities.

See handout

Example 9.3

$\omega = 2\ \text{rads}^{-1}$



$$Z_1 = j1 /\!/ -j/2 = \frac{(j1)(-j/2)}{j1 - j/2} = \frac{1/2}{j/2} = -j1\,\Omega$$

$$Z_2 = j/2 /\!/ 1 = \frac{j/2}{1 + j/2} = \frac{j}{2+j}\cdot\frac{2-j}{2-j} = 0.2 + j0.4\,\Omega$$

At V_1 — remember 'hats'

First $\quad 2(\vec{V}_1 - 5) + \dfrac{\vec{V}_1}{-j} + \dfrac{\vec{V}_1 - \vec{V}_2}{-j} = 0 \quad$ so $\quad 2\vec{V}_1 - 10 + j\vec{V}_1 + j\vec{V}_1 - j\vec{V}_2 = 0$

or $\quad (2 + 2j)\vec{V}_1 - j\vec{V}_2 = 10 \qquad (1)$

Also $\quad \dfrac{\vec{V}_2 - \vec{V}_1}{-j} + \dfrac{\vec{V}_2}{0.2 + j0.4} - 5 = 0$

$\dfrac{1+j2}{5} = 0.2 + j0.4$

so $\quad j\vec{V}_2 - j\vec{V}_1 + \dfrac{5\vec{V}_2}{1+j2}\cdot\dfrac{1-j2}{1-j2} - 5 = 0$

so $\quad j\vec{V}_2 - j\vec{V}_1 + \dfrac{5}{5}\vec{V}_2(1 - j2) - 5 = 0$

so $\quad -j\vec{V}_1 + (1 - j1)\vec{V}_2 = 5 \qquad (2)$

$(2) \times \dfrac{2+2j}{-j}$ i.e. $(2) \times (-2 + j2) \rightarrow (2 + j2)\vec{V}_1 + j4\vec{V}_2 = -10 + j10 \qquad (3)$

$(1) - (3) \rightarrow -j5\vec{V}_2 = 20 - j10$, so $\vec{V}_2 = 2 + j4 = 4.47\angle 63.4°$

Back in (2): $\quad \vec{V}_1 = \dfrac{5 - (1 - j1)(2 + j4)}{-j} = \dfrac{5 - (6 + j2)}{-j} = \dfrac{-1 - j2}{-j} = 2 - j$

So $\quad \vec{V}_1 = 2 - j1 = 2.24\angle -26.6°$

Thus $\quad v_1(t) = 2.24\cos(2t - 26.6°)\ \text{V}$

and $\quad v_2(t) = 4.47\cos(2t + 63.4°)\ \text{V}$

(Note that $v_1(t)$ lags $v_2(t)$ by 90°)

Example 9.3: Nodal Analysis

Step 1: Draw and simplify the equivalent phasor circuit.

Step 2: Write down the nodal equations at v_1 and v_2.

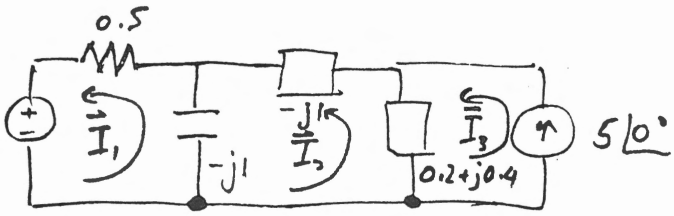
Step 3: Simplify to a pair of simultaneous equations in $\mathbf{V_1}$ and $\mathbf{V_2}$.

Step 4: Solve these by any appropriate method (NB complex numbers!)

Step 5: Convert $\mathbf{V_1}$ and $\mathbf{V_2}$ to polar form, and hence find $v_1(t)$ and $v_2(t)$

Example 9.4

$\omega = 2\,\text{rad s}^{-1}$



$\vec{I}_3 = 5\angle 0^\circ$

By KVL: $-(-j1)(\vec{I}_1 - \vec{I}_2) - 0.5\vec{I}_1 - 5 = 0$ so $(-0.5+j1)\vec{I}_1 - j\vec{I}_2 = 5$ ①

Also $-(0.2+j0.4)(\vec{I}_2 - 5) - (-j1)\vec{I}_2 - (-j1)(\vec{I}_2 - \vec{I}_1) = 0$

so $-j\vec{I}_1 + (-0.2+j1.6)\vec{I}_2 = -1-j2$ ②

① $\times \frac{-0.2+j1.6}{-j}$ i.e. ① $\times (-1.6-j0.2) \rightarrow (-0.5+j1)(-1.6-j0.2)\vec{I}_1 - j(-1.6-j0.2)\vec{I}_2 = 5(-1.6-j0.2)$

so $(1-j1.5)\vec{I}_1 + (-0.2+j1.6)\vec{I}_2 = -8-j1$ ③

② − ③ $\rightarrow (-1+j0.5)\vec{I}_1 = 7-j1$ so $\vec{I}_1 = \frac{7-j1}{-1+j0.5} = \frac{14-j2}{-2+j1} \cdot \frac{-2-j1}{-2-j1}$

$= \frac{30-j10}{5} = -6-j2\text{ A}$ $(\vec{I}_1)$

Back in ①: $(-0.5+j)(-6-j2) - j\vec{I}_2 = 5$

or $(5-j5) - j\vec{I}_2 = 5$

so $\vec{I}_2 = \frac{5-(5-j5)}{-j} = \frac{j5}{-j} = -5\text{ A}$ $(\vec{I}_2)$.

Hence: $\vec{V}_1 = 5 + \frac{1}{2}\vec{I}_1$ in this case

$= 5 + \frac{1}{2}(-6-j2) = 5-3-j1 = 2-j1 = \vec{V}_1$ as found before

Also $\vec{V}_2 = \vec{V}_1 + (-5)(-j) = 2-j+5j = 2+j4$, also as before.

Example 9.4: Mesh Analysis

Step 1: Define mesh currents in the simplified phasor circuit

Step 2: Write down the mesh equations (there are two of them)

Step 3: Simplify to a pair of simultaneous equations in $\mathbf{I_1}$ and $\mathbf{I_2}$.

Step 4: Solve the equations for $\mathbf{I_1}$ and $\mathbf{I_2}$

Step 5: Clearly $\mathbf{V_1} = 5 - \frac{1}{2}\mathbf{I_1}$. Use this to find $\mathbf{V_1}$, and hence find $v(t)$. Compare with the result from nodal analysis.

9.4 Phasor Network Theorems

All of the network theorems that we discussed in covering linear resistive circuits apply also in the case of linear phasor circuits. To illustrate the principle of superposition, let us find the forced response i in Figure 9.3(a). There are two sources, one an ac source with $\omega = 2$ rad/s and one a dc source (where $\omega = 0$). Therefore $i = i_1 + i_2$, where i_1 is due to the ac voltage source acting alone and i_2 is due to the dc current source acting alone. Using an equivalent phasor circuit, we could find i_1 and i_2 by finding the phasor currents $\mathbf{I_1}$ and $\mathbf{I_2}$.

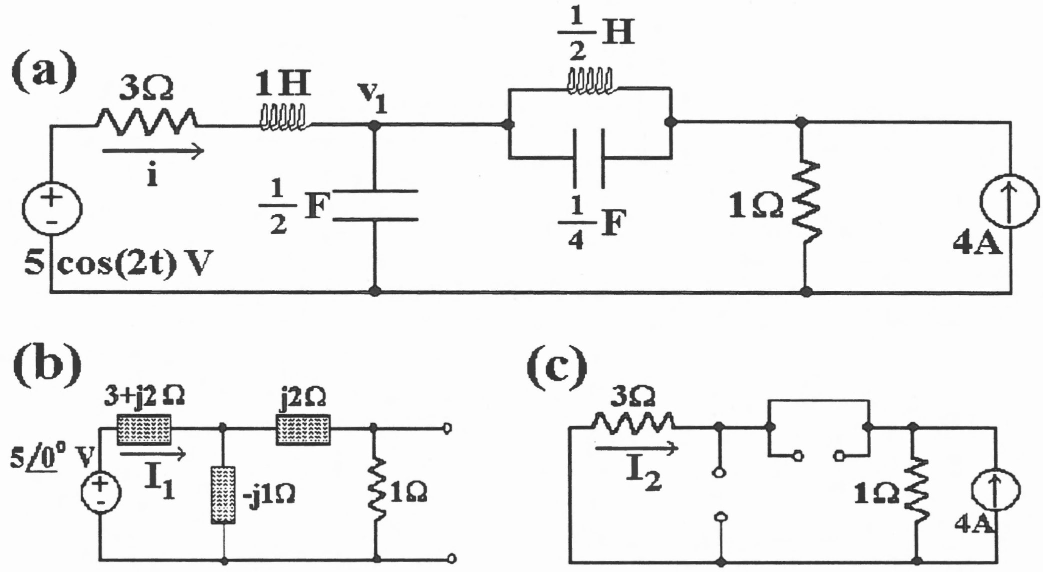


Figure 9.3: The Principle of Superposition: $\mathbf{I} = \mathbf{I_1} + \mathbf{I_2}$

Working first with the ac voltage source, the current source is "killed" by open-circuiting it. This situation is shown in the phasor circuit of Figure 9.3(b). Using impedance combination rules, and noting that the ac source is written $5\angle 0^o$, we get

$$\mathbf{I_1} = \frac{5\angle 0^o}{3 + j2 + [(1 + j2)(-j1)/(1 + j2 - j1)]}$$
$$= \sqrt{2}\angle -8.1^o$$

from which $i_1(t) = \sqrt{2}\cos(2t - 8.1^o)$ A

Then we consider the dc current source. In this case, since $\omega = 0$, all inductors are short-circuited and capacitors are open-circuited, while the ac voltage source is "killed" by short-circuiting it too. The result is the circuit of Figure 9.3(c). The dc current is just 4A, which can be represented by the phasor $4\angle 0^o$, and then current division gives

$$\mathbf{I}_2 = -\frac{1}{1+3}(4) = -1\angle 0^o \text{ A}$$

which is clearly a dc steady state response of $i_2(t) = -1$A. Therefore the total forced response is

$$i(t) = i_1(t) + i_2(t) = \sqrt{2}\cos(2t - 8.1^o) - 1 \text{ A}$$

Note that there is no need for one of the sources to be ac and the other dc. The principle of superposition will hold if all sources are ac, in which case equivalent phasor circuits like Figure 9.3(b) would have to be drawn for each ac source present. Likewise, as we know, all sources may be dc, and our original method of applying the principle of superposition is now seen to be a special case (where $\omega = 0$) of this more general method.

The Theorems of Thèvenin and Norton arise, as we know, from the principle of superposition, and so we find that they too are valid in phasor circuits. The correspondance is very close to the resistive case, the only differences being that the $\mathbf{V}_{oc}$ or $\mathbf{I}_{sc}$ sources are phasors in the ac equivalents, and that we speak of a *Thèvenin equivalent impedance*, Z_{th}, instead of the Thèvenin resistance R_{th}. As you would expect, $Z_{th} = \mathbf{V}_{oc}/\mathbf{I}_{sc}$

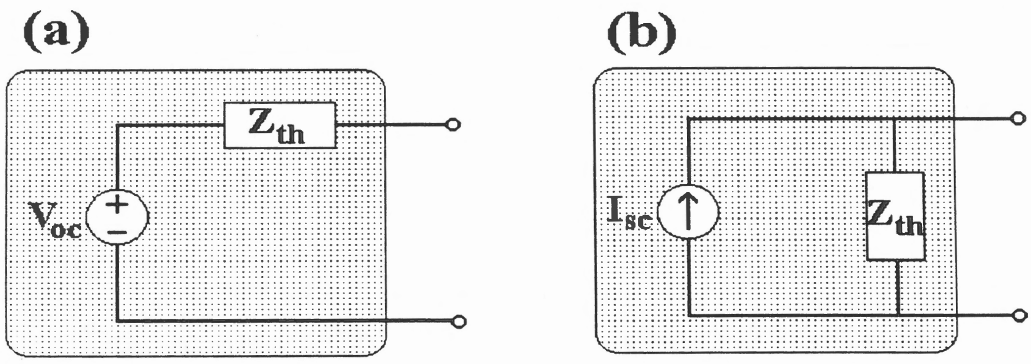


Figure 9.4: Phasor Circuits: (a) Thèvenin Equivalent (b) Norton Equivalent

Finally, since all the phasor circuits that we deal with are *linear* (because they consist only of linear elements), they satisfy the *principle of proportionality.* That is, even though the circuit's sources and voltage and current outputs will be phasors, if all of the sources are multiplied by a scalar factor, then the outputs will be multiplied by the same factor. You will recall that this principle gave us a very useful way to analyse a ladder network in the purely resistive case. The following example shows how this can be extended for circuits that also include inductive and/or capacitive elements.

Example Find the voltage $v(t)$ across the right-hand 1Ω resistor in the circuit below, which is driven at a frequency of 1kHz by the 6V ac source.

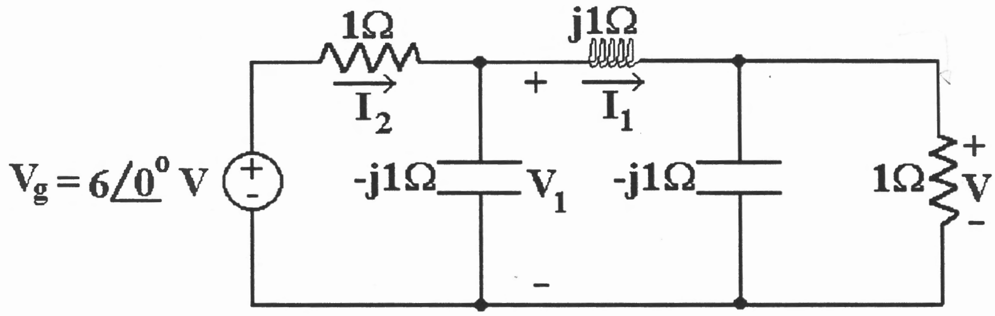


Figure 9.5: A Phasor Ladder Network

We begin by assuming that $\mathbf{V} = 1$ V. Then, from the circuit we have

$$\mathbf{I}_1 = \frac{\mathbf{V}}{1} + \frac{\mathbf{V}}{-j1} = 1 + j1 \text{ A}$$

Then we get:

$$\mathbf{V}_1 = j1\mathbf{I}_1 + \mathbf{V} = j1(1 + j1) + 1 = j1 \text{ V}$$

$$\mathbf{I}_2 = \frac{\mathbf{V}_1}{-j1} + \mathbf{I}_1 = -1 + (1 + j1) = j1 \text{ A}$$

$$\mathbf{V}_g = 1\mathbf{I}_2 + \mathbf{V}_1 = j1 + j1 = j2 \text{ V}$$

So, if $\mathbf{V} = 1$, then $\mathbf{V}_g = j2$. But $\mathbf{V}_g = 6$ V, so we should apply the principle of proportionality and multiply our assumed value for $\mathbf{V}$ by $6/j2$. Hence

$$\mathbf{V} = \frac{6}{j2}(1) = -j3 \text{ V}$$

so

$$v(t) = 3\cos(2000\pi t - 90^o) \text{ V}$$

TUTORIAL 9

9.1

In the circuit of Figure 9.6, find the ac steady-state currents $i(t)$ and $i_1(t)$, using phasors. Sketch both of the currents as functions of time.

$(2\cos(5t - 53.1^o)\mathrm{A};\ 500\cos(5t - 53.1^o)\mathrm{mA})$

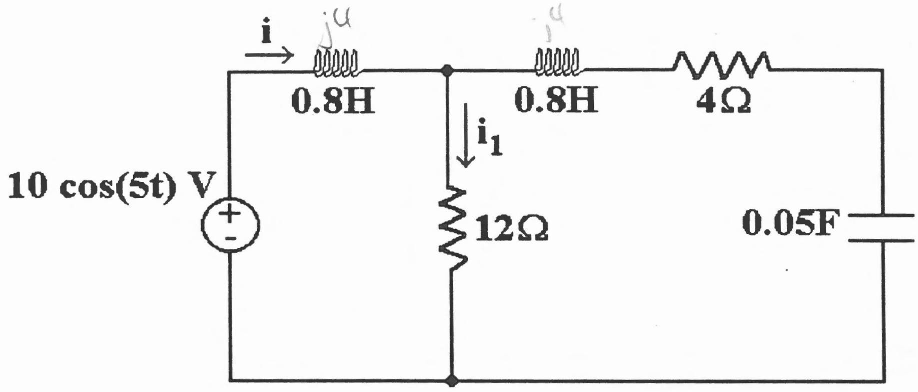


Figure 9.6: Figure for Question 9.1

9.2

In the circuit of Figure 9.7, use the equivalent phasor circuit to find the steady-state value of $v(t)$. Carefully sketch $v(t)$, ignoring transients.

$(30\cos(30000t)\mathrm{mV})$

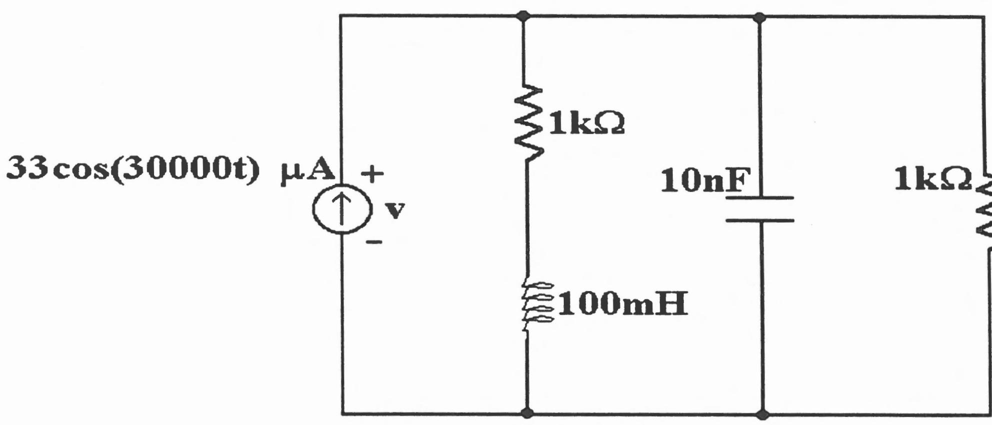


Figure 9.7: Figure for Question 9.2

9.3

Find the steady-state value of v in the circuit of Figure 9.8, using phasors and nodal analysis. Sketch v as a function of time, ignoring transients.

$(15\sqrt{2}\cos(4t - 135^o)\text{V})$

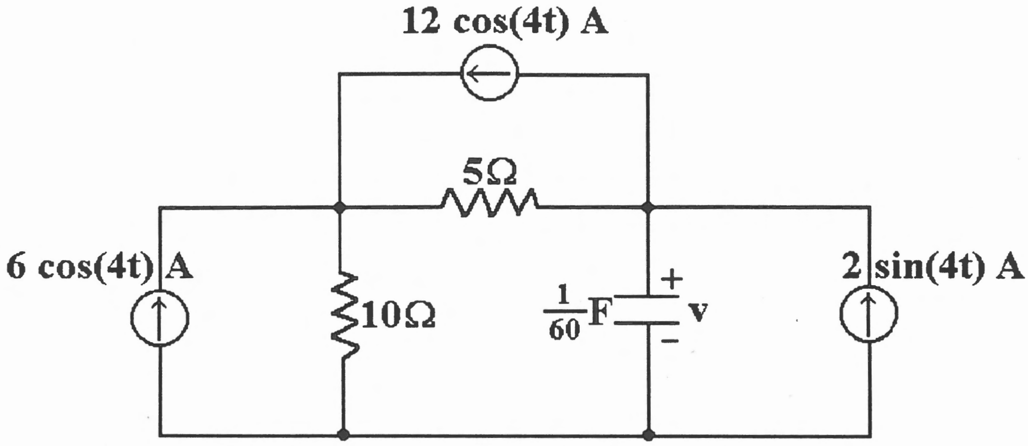


Figure 9.8: Figure for Question 9.3

9.4

Find and carefully sketch the ac steady-state response of $v(t)$ in the circuit of Figure 9.9. Use a phasor equivalent circuit and nodal analysis.

$(35.4\cos(2t - 81.9^o)\ \text{V})$

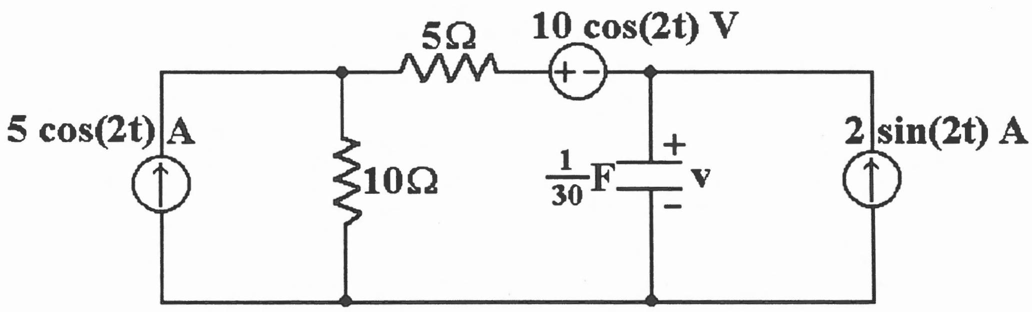


Figure 9.9: Figure for Question 9.4

9.5

Find the ac steady-state voltage v in Figure 9.10, using mesh analysis of the equivalent phasor circuit. Here $i_{g1} = 6\cos(4t)$A and $i_{g2} = 2\cos(4t)$A

($16\sqrt{2}\cos(4t - 45^o)$ V)

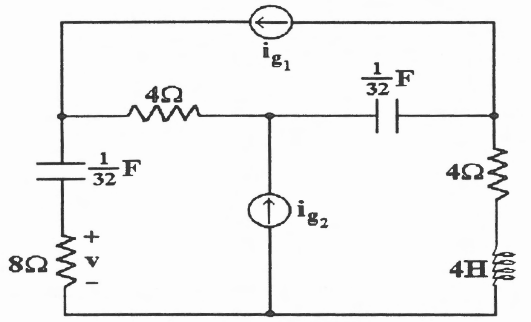


Figure 9.10: Figure for Question 9.5

9.6

(a) Find the ac steady-state current $i(t)$ in Figure 9.11, using mesh analysis in the phasor equivalent circuit.

(b) Draw a graph of the function $i(t)$ that you found in part (a).

(c) Replace the part of the phasor circuit to the left of the terminals a and b with its Thèvenin equivalent, and use the resulting equivalent phasor circuit to calculate the ac steady-state current $i_1(t)$.

(d) State whether your answer to part (c) leads or lags your answer to part (a), and by how much.

($2.83\cos(2t - 45^o)$A; $Z_{th} = 3.61\angle 3.18^o$, $\mathbf{V}_{oc} = 8.05\angle - 26.6^o$, $2\cos(2t)$ A; leads by 45^o)

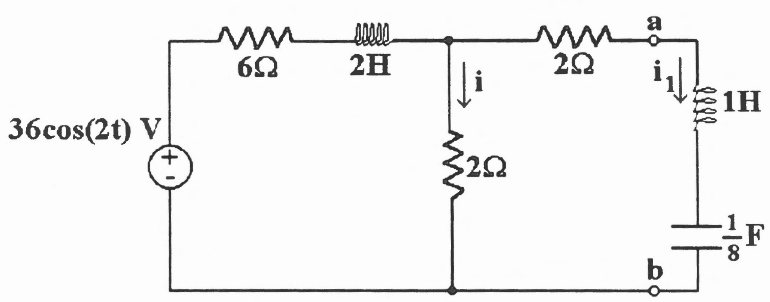


Figure 9.11: Figure for Question 9.6

9.7

In the circuit of Figure 9.12, replace the whole of the circuit apart from the 1Ω resistor by its Thèvenin equivalent, and use the result to find the ac steady-state voltage $v(t)$.

($\mathbf{V}_{oc} = 3.33\angle 0^o$ V; $Z_{th} = -j1.33\Omega$; $2\cos(8t + 53.1^o)$ V)

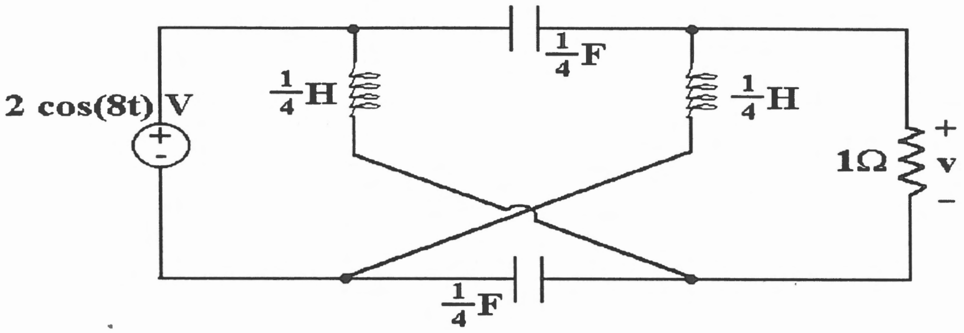


Figure 9.12: Figure for Question 9.7

9.8

(a) Use the principle of superposition in the phasor equivalent to the circuit of Figure 9.13 to find the steady-state value of $v(t)$.

(b) Note that the circuit has a dc source and an ac source. Sketch a graph of your answer to part (a), and distinguish carefully on it between the dc and ac steady-state responses of the voltage v to the combined inputs.

($12{+}1.41\cos(2t - 45^o)$ V)

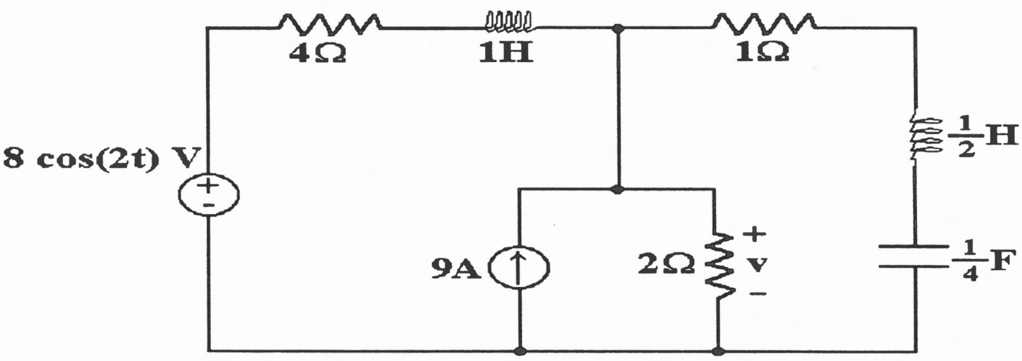


Figure 9.13: Figure for Question 9.8

9.9

(a) Use the principle of superposition in the phasor equivalent of the circuit of Figure 9.14 to find the steady-state value of $v(t)$.

(b) Find an expression for the voltage across the capacitor, $v_c(t)$, if the left side of the capacitor has the positive polarity marking.

$(3\cos(2t) + 12\cos(3t + 7.4^o)$ V)

$(3\cos(2t - 90^o) - 8\cos(3t - 82.6^o)$ V)

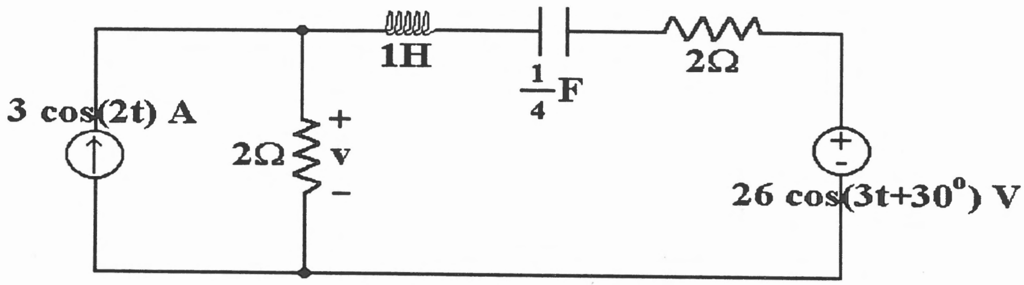


Figure 9.14: Figure for Question 9.9

9.10

Find the ac steady-state voltage $v(t)$ in the circuit of Figure 9.15 by applying the proportionality principle to the corresponding phasor circuit. The voltage source is $v_g = 2\cos(1000t)$ V.

The working is made easier if you begin by assuming that $\mathbf{V} = 100$ *V*

$(\frac{1}{\sqrt{2}}\cos(1000t - 135^o)$ V)

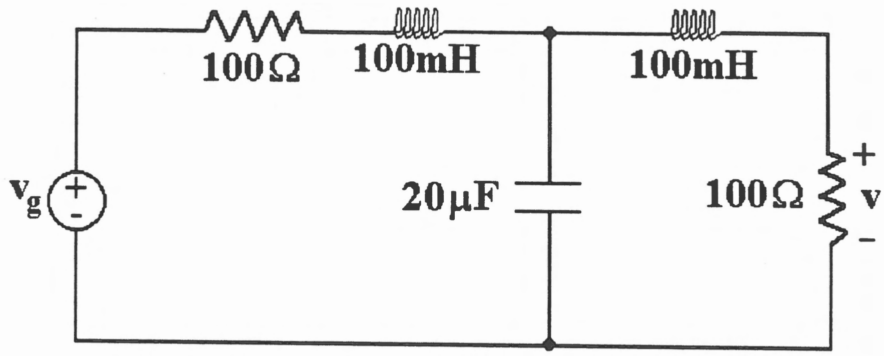
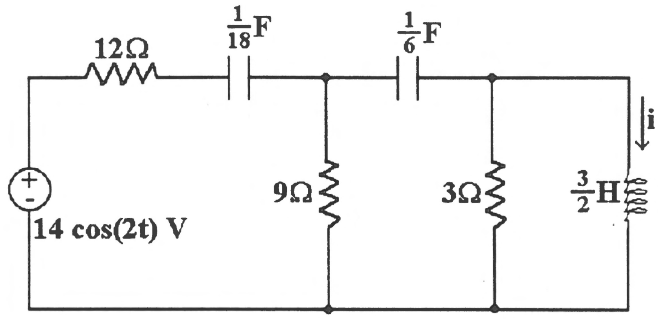


Figure 9.15: Figure for Question 9.10

WORKSHEET 9

9.A

Use the principle of proportionality to find the ac steady state current $i(t)$ in the ladder network shown below. Draw a carefully-labelled phasor equivalent circuit before answering this question. Then you may like to begin by assuming that $i(t) = \cos(2t)$A (i.e. $\mathbf{I} = 1\angle 0^o$).

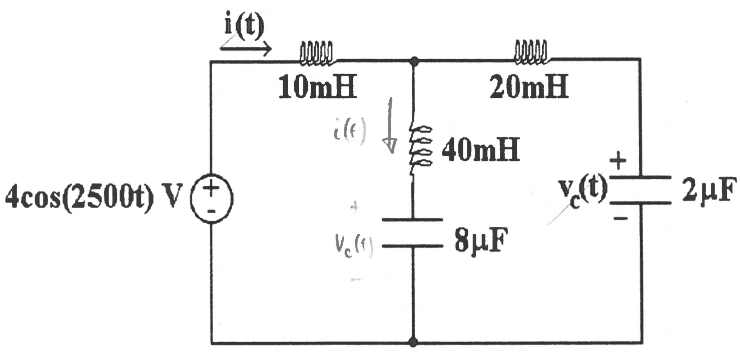


9.B*

(a) Draw the phasor equivalent circuit for the circuit shown below.

(b) Hence, find the ac steady state value of $v_c(t)$, by calculating the equivalent impedance seen by the source, and applying current division.

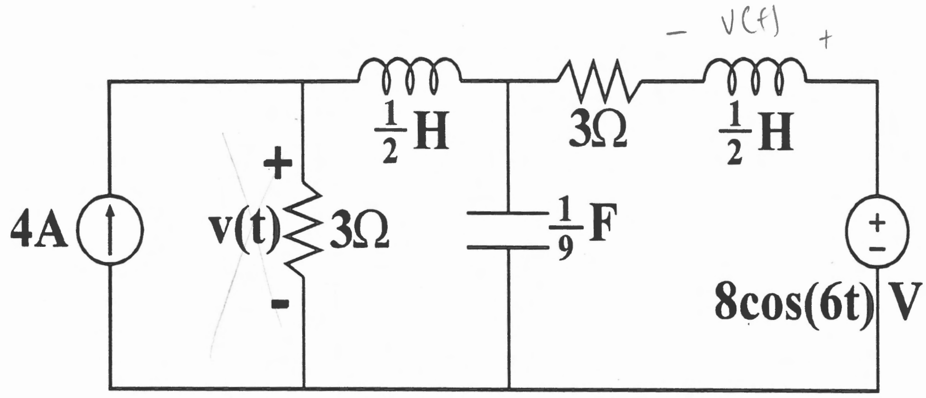
(c) Make a sketch of $i(t)$. Mark your axes clearly to show its behaviour.



9.C*

(a) Explain why a valid method of finding $v(t)$ in the circuit below is to use the Principle of Superposition in the phasor equivalent of the circuit.

(b) Find $v(t)$, using the method suggested in part (a).



9.D

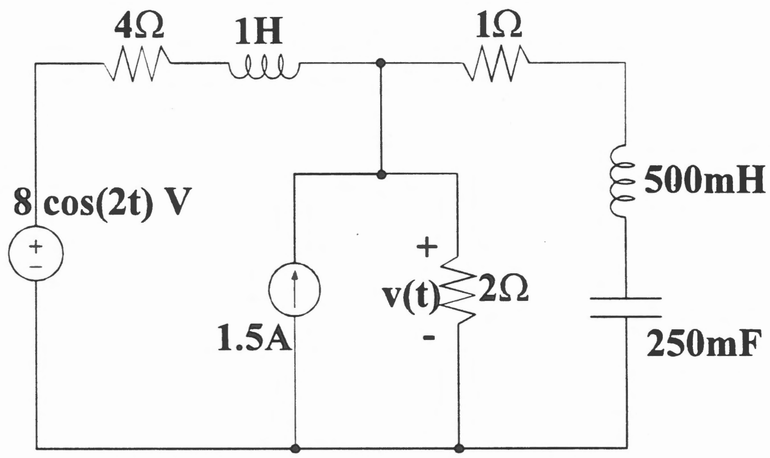
Four resistors, each of resistance 1kΩ, are connected so that each one forms one side of a square, $ABDC$. An ac voltage generator of amplitude 12V and of frequency 1kHz is attached to point A, while point C is connected to a 12V dc voltage source. Both of these sources are referenced to point D, which is at ground potential.

(a) Find the voltage at point B, referencing its magnitude to point D and its phase to the phase of the ac voltage generator;

(b) Repeat part (a) for the case where the 1kΩ resistor joining points A and B is replaced with a 100nF capacitor.

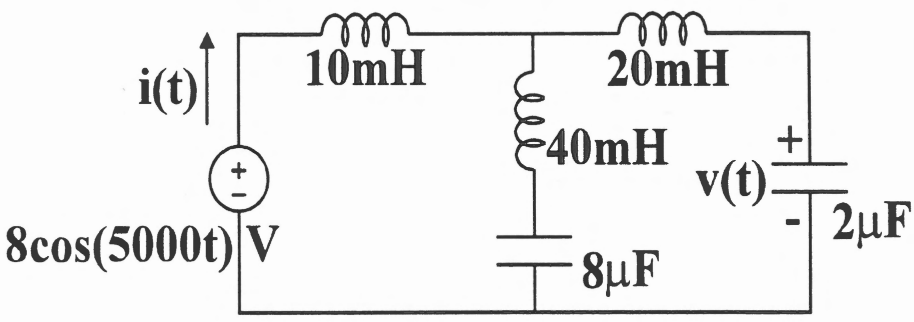
9.E

(a) Draw the phasor equivalent circuit, referred to the ac voltage source, for the circuit shown below.
(b) Find the ac steady state value of $v(t)$, by making use of the Principle of Superposition in your equivalent circuit.
(c) Draw the graph of $v(t)$. Mark your axes clearly with all relevant quantities to show its behaviour unambiguously.



9.F

Draw the phasor equivalent circuit to the circuit shown below. Hence, find the ac steady state values of $i(t)$ and $v(t)$, by calculating the equivalent impedance seen by the source, and applying current division.



Chapter 10

AC Steady-State Power

In Lectures C22 and C23 we cover:

- The concepts of *instantaneous* and *average* ac power
- Power and the Principle of Superposition
- The use of *RMS values*
- A measure relating the average power to the power that is apparently available, known as the *power factor*
- An important industrial technique, called *power factor correction*

10.1 Instantaneous and Average AC Power

We begin by defining the *instantaneous* power in an element with voltage $v(t)$ and current $i(t)$ to be

$$p(t) = v(t)i(t)$$

If $v(t)$ and $i(t)$ are both periodic functions of identical period, then $p(t)$ is also periodic, and has the same period. Starting with the periodicity of $v(t)$ and $i(t)$, we can easily see that this is true:

$$v(t+T) = v(t) \qquad \text{and} \qquad i(t+T) = i(t)$$

so $$p(t+T) = v(t+T)i(t+T) = v(t)i(t) = p(t)$$

However, note that the *fundamental* period of the power, written T_1, is not necessarily equal to T, the period of the voltage and current. Instead, $T = nT_1$, where n is an integer. For example, if current $i(t) = I_m \cos(\omega t)$ flows in a resistor R, in which case $T = \frac{2\pi}{\omega}$, then

$$
\begin{aligned}
p(t) &= Ri^2(t) \\
&= RI_m^2 \cos^2(\omega t) \\
&= \frac{RI_m^2}{2}(1 + \cos(2\omega t))
\end{aligned}
$$

which clearly has a period of $T_1 = \frac{\pi}{\omega}$, so that $T = 2T_1$ (i.e. $n = 2$).

On the other hand, if $i(t) = I_m(1 + \cos(\omega t))$, which has $T = \frac{2\pi}{\omega}$, then the power in the resistor is

$$
\begin{aligned}
p(t) &= RI_m^2(1 + \cos(\omega t))^2 \\
&= RI_m^2(2 + 2\cos(\omega t) + \cos^{2}(\omega t))
\end{aligned}
$$

$RI_m^2(2 + 2\cos(\omega t) + \frac{1}{2} + \frac{1}{2}\cos(2\omega t)$

which (if graphed) will be found to have period $T_1 = \frac{2\pi}{\omega}$, so that in this case $T = T_1$ (i.e. $n = 1$).

$RI_m^2(\frac{3}{2} + 2\cos(\omega t) + \frac{1}{2}\cos(2\omega t))$

fundamental frequency

We now further define the *average power* to be the mean value of the instantaneous power, taken over a cycle of the periodic power function. We can use integration, starting at arbitrary time t_1, to obtain the expression for average power (note the capital letter):

$$P = \frac{1}{T_1}\int_{t_1}^{t_1+T_1} p(t)\, dt$$

Note that we can just as well obtain the average power by integrating over *any whole number* of cycles of the power, including over T. The unit of average power, as for instantaneous power, is the *watt* (W).

As we are often called upon to integrate power functions over the period $\frac{2\pi}{\omega}$, the table on the next page is useful for the calculation of average powers. Note that m and n are integers in the expressions to be integrated.

Now let us take a closer look at average power in a two-terminal device. Let the device have impedance

$$Z = |Z|\angle\theta^\circ$$

Useful Power Integrals		
No	$f(t)$	$\int_0^{2\pi/\omega} f(t)\,dt, (\omega \neq 0)$
1.	$\sin(\omega t + \alpha)$	0
2.	$\cos(\omega t + \alpha)$	0
3.	$\sin(n\omega t + \alpha)$	0
4.	cos $\sin(n\omega t + \alpha)$	0
5.	$\sin^2(\omega t + \alpha)$	π/ω
6.	$\cos^2(\omega t + \alpha)$	π/ω
7.	$\sin(m\omega t + \alpha)\cos(n\omega t + \alpha)$	0
8.	$\cos(m\omega t + \alpha)\cos(n\omega t + \beta)$	$0\ (m \neq n)$ $\pi\cos(\alpha - \beta)/\omega\ (m = n)$

and let the device be driven by a general voltage

$$v(t) = V_m \cos(\omega t + \phi)$$

In response to this voltage it is clear from our work with phasors that the current that flows can be written

$$i(t) = I_m \cos(\omega t + (\phi - \theta))$$

where $I_m = V_m/|Z|$, since $\mathbf{I} = \mathbf{V}/Z$.

We can now use our table of integrals to work out the average power:

$$P = \frac{1}{T}\int_0^{2\pi/\omega} v(t)i(t)\,dt$$

$$= \frac{\omega}{2\pi}V_m I_m \int_0^{2\pi/\omega} \cos(\omega t+\phi)\cos(\omega t+\phi-\theta)\,dt$$

$$= \frac{V_m I_m}{2}\cos(\theta) \qquad \text{(by 8 with } m = n)$$

Note that θ is the *argument* of the impedance, or it is the angle by which the voltage $v(t)$ across the load *leads* the current $i(t)$ through the load.

$Z\angle\theta$

Cap : $\theta = -90°$

L = $\theta = 90°$ $P = 0$

R = $\theta = 0$ $P = \frac{V_m I_m}{2}$

AC

Now, if the two-terminal device is a resistor, then $\theta = 0$, since $Z_R = |Z|\angle 0^o$. Furthermore, $V_m = RI_m$ by Ohm's Law, so the average power dissipated in a resistor in the ac steady state (i.e. with ac currents and voltages present) is

$$P_R = \frac{1}{2}V_m I_m = \frac{1}{2}RI_m^2$$

Contrast this with the power dissipated in the resistor if the source is dc, which is RI_{dc}^2 (without the $\frac{1}{2}$). If, on the other hand, the two-terminal device is an inductor or a capacitor, then $\theta = 90^o$ or $\theta = -90^o$ respectively, and so in either case $P = 0$. So here we have confirmation that there is no power dissipated in a capacitor or in an inductor: these elements store energy temporarily for later release, but they are (ideally) *lossless.*

Another useful expression for average power comes from the theory of complex numbers. We know that we can write the impedance as

$$z = |Z|\angle\theta = \Re(Z) + j\Im(Z) = |Z|\cos\theta + j|Z|\sin\theta$$

and so
$$\cos\theta = \frac{\Re(Z)}{|Z|}$$

and hence we can write the average power as

$P = \frac{V_m I_m}{2} \cos\theta$ (see on) $\frac{V_m}{|Z|} = I_m$

$$P = \frac{1}{2}I_m^2(\Re(Z)).$$

As an example, consider the power delivered by the source in this circuit:

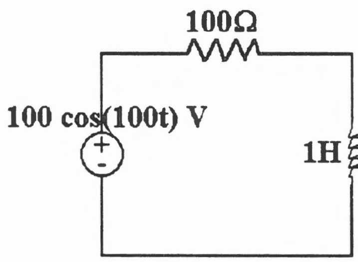


Figure 10.1: An AC Source Driving an Inductive Load

The source sees impedance

$$Z = 100 + j100\ \Omega \; = \; 100\sqrt{2}\angle 45^\circ\ \Omega$$

and $V_m = 100$ V, so

$$I_m = \frac{V_m}{|Z|} = \frac{100}{100\sqrt{2}} = \frac{1}{\sqrt{2}}\ \text{A}$$

and so $$P = \frac{1}{2}V_m I_m \cos(\theta) = \frac{100}{2\sqrt{2}}\cos 45^\circ = 25\ \text{W}$$

Alternatively, tackle the problem using our most recent power equation:

$$P = \frac{1}{2}I_m^2(\Re(Z)) = \frac{1}{2} \times \frac{1}{2} \times 100 = 25\ \text{W (as before).}$$

Note that the power absorbed by the 100Ω resistor is

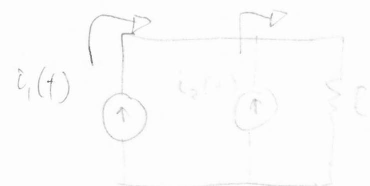
$$P_R = \frac{RI_m^2}{2} = \frac{100(\frac{1}{\sqrt{2}}^2)}{2} = 25\ \text{W}$$

which we interpret as meaning that all of the power delivered to the source is dissipated in the load resistor, and none of it is lost in the inductor.

10.2 Superposition and Power

If there is more than one source in a circuit, you should be aware that superposition does *not* apply, if you are trying to find the instantaneous power supplied to a given element in the circuit. This is because power is a *square* function of current or voltage, and not a linear function. If, for example, two sources supply currents $i_1(t)$ and $i_2(t)$, then the instantaneous power in a resistor R is

$$\begin{aligned} p(t) &= R(i_1(t) + i_2(t))^2 \\ &= Ri_1^2(t) + Ri_2^2(t) + 2Ri_1(t)i_2(t) \\ &= p_1(t) + p_2(t) + 2Ri_1(t)i_2(t) \\ &\neq p_1(t) + p_2(t) \end{aligned}$$



If, however, both sources are periodic (including the dc case where $\omega = 0$) then superposition *does* apply to the *average power*, but only if the sources

are *orthogonal.* To see what this means, we calculate the average power in the above example as

$$
\begin{aligned}
P &= \frac{1}{T}\int_0^T p(t)\,dt \\
&= \frac{1}{T}\int_0^T (p_1(t) + p_2(t) + 2Ri_1(t)i_2(t))\,dt \\
&= P_1 + P_2 + \frac{2R}{T}\int_0^T i_1(t)i_2(t)\,dt
\end{aligned}
$$

Thus $P = P_1 + P_2$ (i.e. superposition holds) only if

$$\int_0^T i_1(t)i_2(t)\,dt = 0$$

in which case we say that $i_1(t)$ and $i_2(t)$ are *orthogonal.* You will meet this important condition again in later courses, but for now you should know that it holds if $i_1(t)$ and $i_2(t)$ are any two sinusoids of *different* frequencies, but it does not hold if the two sources have the same frequency. You can see that this is so from the table of integrals included in these notes. In the latter case, if we wanted to know the average power consumed in an element of the circuit, we would use superposition to find the combined *current* flowing in the element (superposition holds for this, because the element is linear), and then we would use the current to find the power.

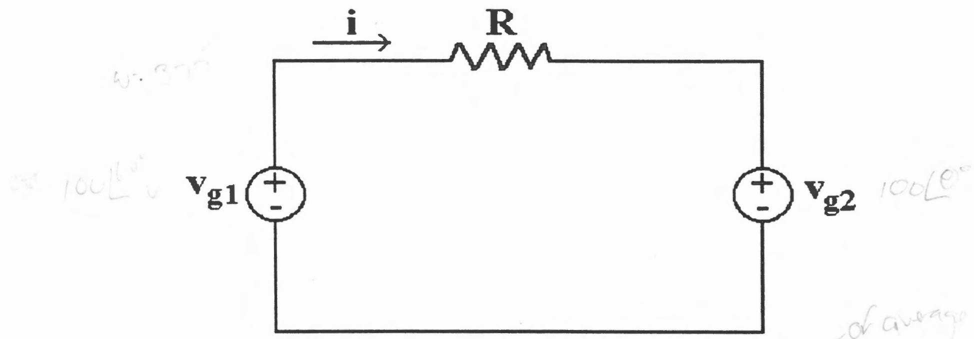
Example 10.1

With reference to the circuit in the Figure on the next page:

(a) Find the average power dissipated in the resistor if $R = 100\Omega$, $v_{g1} = 100\cos(377t + 60^o)$V and $v_{g2} = 50\cos(377t)$V.

(b) Find the average power dissipated in the 100Ω resistor if v_{g2} is now replaced with a 50V dc source.

10.1 Superposition Example



(a) 2 sources have same frequency ∴ superposition of average power does not apply

$\vec{I}_1 = \frac{\vec{V}_{g1}}{R} = \frac{100\angle 60^\circ}{100} = 1\angle 60^\circ = \cos(60) + j\sin(60) = \frac{1}{2} + j\frac{\sqrt{3}}{2}$

$\vec{I}_2 = -\frac{\vec{V}_{g2}}{R} = \frac{-50\angle 0^\circ}{100} = -\frac{1}{2}\,A$

we can add these two complex numbers bcs they have the same frequency

$\vec{I} = \vec{I}_1 + \vec{I}_2 = \frac{\sqrt{3}}{2}j$

$= \frac{\sqrt{3}}{2}j\cos(377t + 90^\circ)$

$P = \frac{R I_m^2}{2} = \frac{100}{2} \times \left(\frac{\sqrt{3}}{2}\right)^2 = 37.5W$

(b) Sources have diff freq ∴ Superposition of avg power does apply

$\vec{I}_1 = 1\angle 60^\circ \qquad \vec{I}_2 = \frac{1}{2}A$

$P_1 = \frac{R I_{max,1}^2}{2} = \frac{100 \times 1^2}{2} = 50W$

$P_2 = R I_{2max}^2$ — dc – no 2

$= 100 \times \frac{1}{4}$

$= 25W$

Average power $= P_1 + P_2 = 75W$

by changing from AC source of max 50 to dc of 50V we double the power

10.3 RMS Values

We have seen that periodic sources deliver an *average power* to a resistive load. If we want to compare the powers delivered by different waveforms, we often measure them by the dc current or voltage that would deliver the same average power to a resistance R. So we equate (in the case of current)

$$P = RI_{dc}^2 = \frac{1}{T}\int_0^T Ri^2(t)\,dt$$

from which

$$I_{dc} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}$$

Since the current waveform has been squared, meaned and rooted to get to the measure that we want, we call I_{dc} the *root-mean-square* or *rms* value of $i(t)$ and write it I_{rms}. We can also speak of V_{rms} for voltage waveforms by a similar development. Note that rms values can be calculated for *any* periodic waveform, and that for dc currents or voltages the rms value is, of course, just the dc value of the function.

Consider a general sinusoidal current $i(t) = I_m \cos(\omega t + \phi)$. Since the period is $T = \frac{2\pi}{\omega}$, we can calculate the rms current as

$$I_{rms} = \sqrt{\frac{\omega I_m^2}{2\pi}\int_0^{2\pi/\omega} \cos^2(\omega t + \phi)\,dt}$$

which, using entry number 6 in the table of integrals above, leads to

$$I_{rms} = \frac{I_m}{\sqrt{2}}$$

One can also show, by an identical development, that

$$V_{rms} = \frac{V_m}{\sqrt{2}}.$$

We can now re-write the two main formulae for average power, to get them in terms of the rms quantities:

$$P = V_{rms} I_{rms} \cos(\theta) = I_{rms}^2 \Re(Z)$$

These relationships are extremely useful, particularly in the field of power generation and distribution, where rms quantities are very frequently seen.

Many household appliances are intended for use with a 220V 50Hz source, which is generally available as mains electricity. The peak value of this sinusoidal voltage is $220\sqrt{2} \approx 311$V, because the quoted 220V is actually the rms value. Furthermore, if the appliance is described as drawing 13A, then this will also be an rms value, so the average power which the device consumes will be

220 2.86kW

$$P = V_{rms} I_{rms} \cos(0^o) = 240V \times 13A = 3.12\text{kW}$$

if it is a purely resistive load. However, this will be modified by $\cos(\theta)$ if the load ($Z_L = |Z|\angle\theta^o$) has capacitive or inductive components contributing to its impedance. A classic example of an inductive load would be a motor.

10.4 The Power Factor

This "modification" by $\cos(\theta)$ of the power supplied to a pure resistance gives rise to some additional useful power definitions. We have seen that the average power is

$$P = V_{rms} I_{rms} \cos(\theta)$$

The product $V_{rms} I_{rms}$, which is easy to measure by finding V_{rms} and I_{rms} using a meter, is called the *apparent power*, and is measured in units of *voltamperes* (VA), or, more frequently, in *kilovoltamperes* (kVA). The ratio of average power to apparent power is defined as the *power factor*, so we have

$$pf = \frac{P}{V_{rms} I_{rms}} = \cos(\theta)$$

where θ, the argument of the load impedance, is often also called the *PF angle*.

You can now easily see that, in the case of purely resistive loads, the voltage

and current are in phase, so $\theta = 0$ and $pf = 1$. Thus, average power equals apparent power. This is also the case if the load has capacitance and inductance, but only if the capacitive and inductive reactances cancel one another.

In a purely reactive load (pure C or pure L) $\theta = \pm 90^o$ and $pf = 0$ and as we have seen, the average power is zero. Realistically, however, all loads have *some* resistance, and so we generally encounter RC or RL combinations in a load. For RC combinations, $-90^o < \theta < 0^o$, and $\cos(\theta)$ is described as a *leading pf*, whereas RL combinations have $0^o < \theta < 90^o$, and $\cos = (\theta)$ is called a *lagging pf.* You will note that the terms used also describe how the phase of the current through the load behaves with respect to the phase of the voltage across it.

As an example, consider a load consisting of a 100Ω resistor in series with a 0.1H inductor, driven at 50Hz. Clearly the impedance is

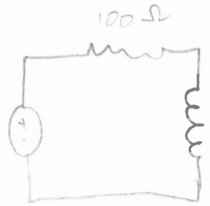
$$Z = 100 + j31.4\ \Omega \ = \ 104.8\angle 17.44^o$$

and so the load has a pf of $\cos(17.44^o)$ or <u>0.954 lagging</u>.

In industrial applications, where loads may require many *kilo*watts of power, the power factor greatly affects the electric bill. We see this in terms of the following example, from which you should be able to appreciate that loads which operate at lower power factors cost the generating company *more* to supply a given amount of average power. You will see from the example why generating companies charge out their electricity according to the *apparent power* consumed, rather than by the user's average power.

50 Hz

$\omega = 2\pi f = 100\pi$ rad/s



$\leftarrow 100\pi = j31.42\ \Omega$

$Z = 100 + j31.42\ \Omega$

$= 104.8\angle 17.44^\circ$

$pf = \cos 17.44^\circ$

$= 0.954$ lagging

Example 10.2

A mill consumes an average power of 100kW from a 220V (rms) line. Find the rms current into the mill, and the apparent power that it consumes, if it is operating at a pf of 0.85 lagging.

$I_{rms} = \frac{P_{avg}}{V_{rms}\cos\theta} = \frac{100\text{ kW}}{220 \times 0.85} = 534.8\text{ A rms}$

$AP = V_{rms} \times I_{rms} = 220 \times 534.8\text{ A} = 117.65\text{ kVA}$

If the pf is somehow increased to 0.95 lagging, find the new rms current drawn and the apparent power now consumed.

$I_{rms} = \frac{P}{V_{rms}\cos\theta} = \frac{100k}{220 \times 0.95} = 478.5\text{ A rms}$ (10% low)

$AP = 105.26\text{ kVA}$

If the generator supplying the power has a resistance of 0.1Ω, find out the average power that it must generate in each of the above cases in order to supply 100kW to the mill. If *you* ran the generating company, how would you charge out the power?

power dissipated in own resistance

Supply $= 100\,000 + i^2R = 128.6\text{kW}$

$(534.8)^2 \quad 0.1$

power to him

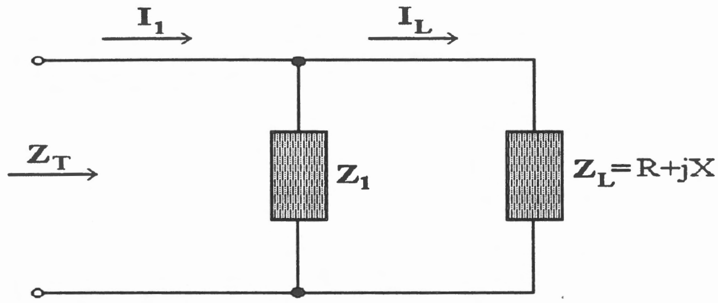
on basis of Apparent power

Supply $= 100\,000 + i^2R = 122.9\text{kW}$ [6% low]

add capacitor, to add so lead to load

$Z_c = R + jX$

The preceding example should motivate industrial consumers of electricity to try to keep the power factor at which they operate as close to *unity* as possible. We now consider how this can be done by connecting an impedance Z_1 *in parallel with* the load $Z_L = r + jX$. The situation is shown in the Figure. When Z_1 is added, since Z is fixed, I does not change, but the current



supplied by the generator, I_1, does change. Let us denote the impedance of the parallel combination by

$$Z_T = \frac{Z_L Z_1}{Z_L + Z_1}$$

In general, we select Z_1 so that (1) Z_1 absorbs zero average power, and (2) Z_T satisfies some new desired power factor which we call $pf = PF$. The first condition requires that Z_1 be purely reactive. That is

$$Z_1 = jX_1$$

The second condition requires that

$$\cos\left[\tan^{-1}\left(\frac{\Im(Z_T)}{\Re(Z_T)}\right)\right] = PF$$

Substituting Z_T in terms of R, X and X_1 into this equation, you should be able to show that

$$X_1 = \frac{R^2 + X^2}{R\tan(\cos^{-1}(PF)) - X}$$

learn for exam

where we note that $\tan(\cos^{-1}(PF))$ is positive if PF is lagging and negative if PF is leading. If the desired power factor is unity, then the above equation simplifies, and the parallel impedance to be added is just

$$X_1 = -(R^2 + X^2)/X$$

$\frac{-j}{\omega C} = X_1$

TUTORIAL 10

10.1

(a) Use the defining formulas for instantaneous power and average power

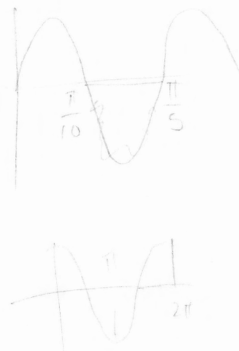
$$p(t) = v(t)i(t) \qquad \text{and} \qquad P = \frac{1}{T}\int_{t1}^{t1+T} p(t)\,dt$$

along with the table of integrals provided in the notes, to verify that the average power absorbed by a capacitor of C farads carrying a current

$$i(t) = I_m \cos(\omega t)$$

is zero. Repeat this for an inductor of L henrys.

(b) Find the average power delivered to a 10Ω resistor carrying a current of



(i) $i(t) = 5|\sin(10t)|$ mA

(ii) $i(t) = \begin{cases} 10\sin(10t)mA & (0 \le t < \pi/10s) \\ 0 & (\pi/10 \le t < \pi/5s) \end{cases}$

and $T = \pi/5$ s

(125μW; 250μW)

10.2
(a) In the circuit of Figure 10.2, find the average power absorbed by the capacitor, by each of the two resistors, and by the source.

(b) Find the average power delivered to a 10Ω resistor carrying a current of

$$\text{(i) } i(t) = \begin{cases} 6mA & (0 \leq t < 10ms) \\ 0 & (10ms \leq t < 20ms) \end{cases}$$

and $T = 20$ ms

$$\text{(ii) } i(t) = \; 3tA \quad (0 \leq t < 4s)$$

and $T = 4$ s

(0, 13.3W, 2.67W, -16.0W; 180μW, 480W)

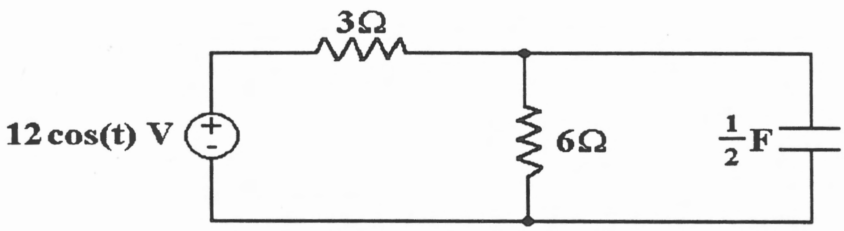


Figure 10.2: Figure for Question 10.2

10.3
Find the average power delivered to the resistor in Figure 10.3, if $R = 20\Omega$ and

(a) $v_{g1} = 20\cos(100t)$ V and $v_{g2} = 10\cos(100t + 60^o)$ V
(b) $v_{g1} = 20\cos(t + 60^o)$ V and $v_{g2} = 100\sin(2t - 30^o)$ V
(c) $v_{g1} = 50\cos(t + 30^o)$ V and $v_{g2} = 100\sin(t + 60^o)$ V
(d) $v_{g1} = 20\cos(t + 25^o)$ V and $v_{g2} = 30\sin(5t - 35^o)$ V

(7.5W; 260W; 188W; 32.5W)

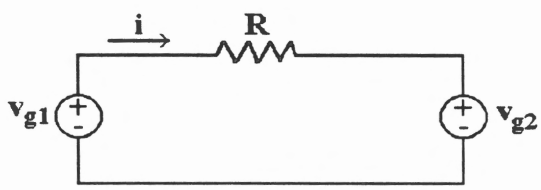


Figure 10.3: Figure for Question 10.3

10.4

(a) In the circuit of Figure 10.4(a), find the average power absorbed by each resistor and by each source.

(b) In the circuit of Figure 10.4(b), find the average power absorbed by each resistor and by each source.

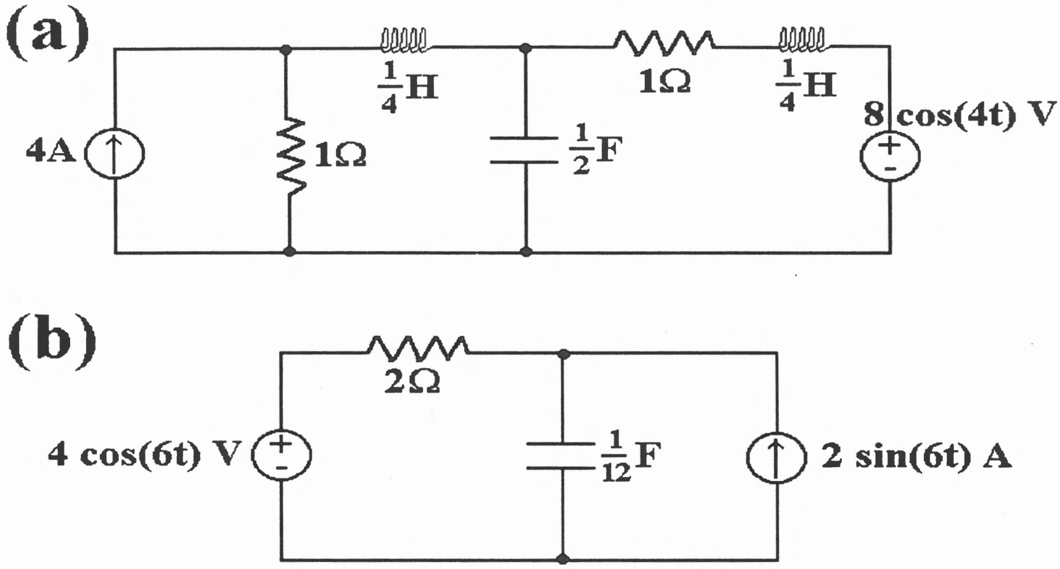


Figure 10.4: Figures for Question 10.4

(8W, 24W, -8W, -24W; 4W, -2W, -2W)

10.5

Find the rms value of a periodic current for which one period is defined by

(a) $i(t) = \begin{cases} I_m & (0 \leq t < 2s) \\ -I_m & (2 \leq t < 4s) \end{cases}$

(b) $i(t) = \ 2t \quad (0 \leq t < T)$

(c) $i(t) = \ I_m \sin(\omega t) \quad (0 \leq t \leq \pi/\omega) \qquad$ and $T = \pi/\omega$

$(I_m, \ 2T/\sqrt{3}, \ I_m/\sqrt{2})$

10.6
Find the rms value of

(a) $i(t) = 10\cos(\omega t) + 20\sin(\omega t - 30^o)$ A

(b) $i(t) = 8\cos(\omega t - 20^o) + 6\sin(2\omega t + 10^o)$ A

(c) $i(t) = I(1 + \cos(377t))$ A

(12.25A, 7.07A, $I\sqrt{\frac{3}{2}}$A)

10.7
Find V_{rms} in the circuit of Figure 10.5.
(2V)

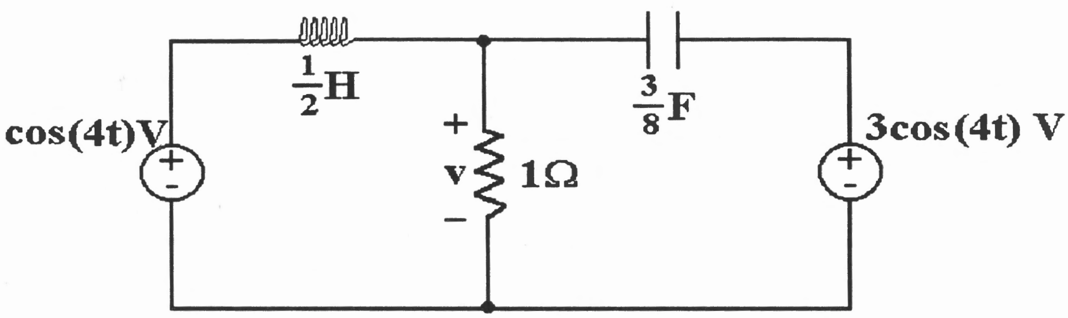


Figure 10.5: Figure for Question 10.7

10.8
Find the apparent power for

(a) a load that requires 20A rms from a 115V rms line

(b) a load consisting of a 100Ω resistor in parallel with a 25μF capacitor, connected to a 120V 60Hz source.

(2.3kVA; 197.9VA)

10.9
Find the power factor for

(a) a load consisting of a series connection of a 10Ω resistor and a 10mH inductor, operating at 60Hz.

(b) a capacitive load requiring 25A rms and 5kW at 230V rms

(c) a load that is a parallel connection of a 5kW load with a power factor of 0.9 leading, and a 10kW load with a 0.95 lagging power factor.

(0.936 lagging; 0.87 leading; 0.998 lagging)

10.10
The circuit of Figure 10.6 represents a motor which is in use at a certain factory. You are asked to advise the factory manager as to the value of a shunt capacitance which should be connected across his motor in order to achieve the following power factors:

(a) 0.95 leading

(b) 0.85 lagging

(c) unity

(66.4μF; 19.0μF; 50μF)

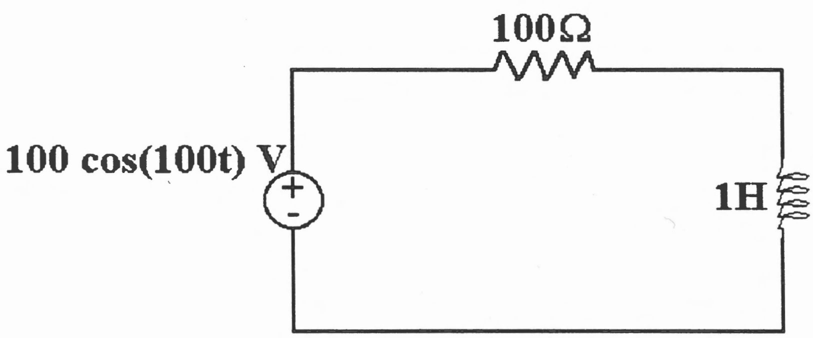
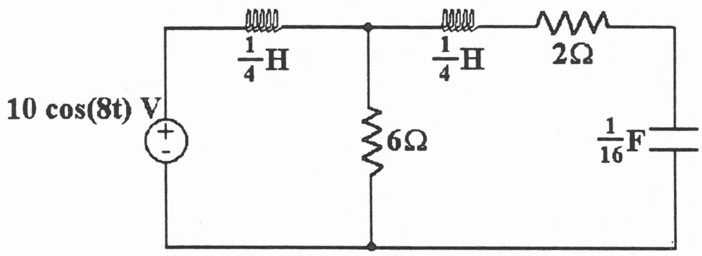


Figure 10.6: Figure for Question 10.10

WORKSHEET 10

10.A

Find the power factor seen from the terminals of the source in the Figure below, and the reactance necessary to connect in parallel with the source to change the power factor to unity.



10.B*

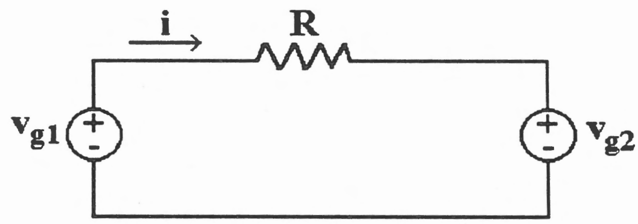
(a) Find the average power delivered to the resistor in circuit (a) below, if $R = 10\Omega$ and

(i) $v_{g1} = 20\cos(100t)$ V and $v_{g2} = 10\cos(100t + 60^o)$ V

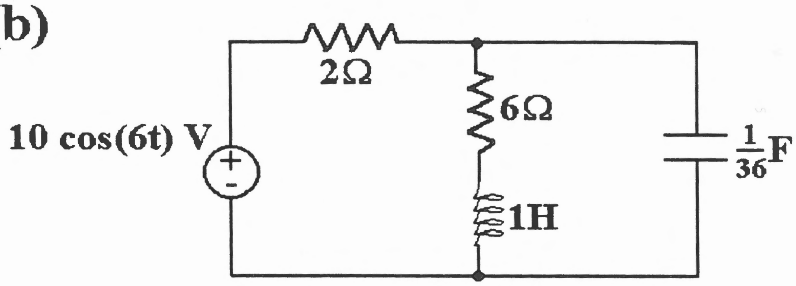
(ii) $v_{g1} = 20\cos(t + 60^o)$ V and $v_{g2} = 100\sin(2t - 30^o)$ V

(b) Find the reactive element required to be connected across the voltage source in circuit (b) below in order to make the power factor seen by the source equal to unity.

(a)



(b)

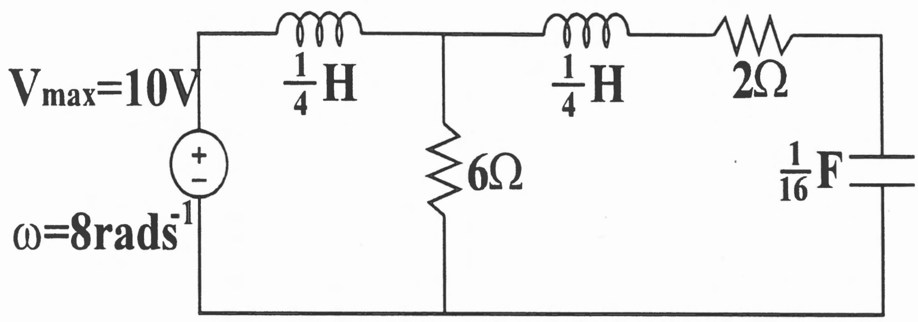


10.C*

(a) State the rms value of the voltage $v(t) = 10\cos(8t)$V

(b) Find the power factor seen from the terminals of the voltage source in the circuit below.

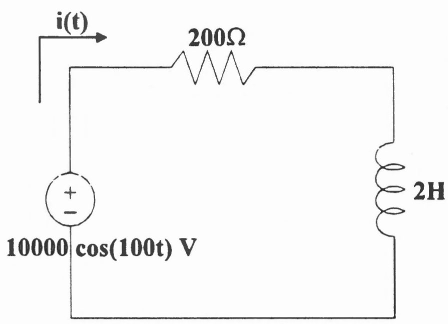
(c) What value of shunt capacitance should be connected in parallel with the voltage source to change the power factor to unity?



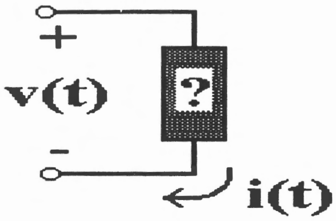
10.D

The circuit below represents a motor which is in use at a certain factory. You are asked to advise the factory manager as to the following:

(a) What is the power factor angle (θ in degrees) represented by the load, and hence what is the power factor of the load?
(b) What current, $i(t)$, is drawn by the load and, hence, what is the apparent power consumed by the factory?
(c) What is the average power consumed by the factory, and what value of shunt capacitance will make equal quantities of the average power and apparent power?



10.E



The element shown has voltage $v(t)$ across it and current $i(t)$ through it, where:

$$v(t) = 2\sin(t) \qquad \text{and} \qquad i(t) = \cos(t)$$

(a) Find and simplify an expression for the instantaneous power $p(t)$ that is absorbed by the element.

(b) Show that the instantaneous power oscillates with twice the frequency of the current or the voltage.

(c) What is the average power absorbed by the element?

(d) State the rms values of voltage and current.

(e) Make a neat sketch, on a single pair of axes, of $v(t)$ and $i(t)$.

(f) Use your sketch to state what kind of component (resistor, capacitor or inductor) the given element is.

(g) Now state the value of the given element, with units.

UNIVERSITY OF CAPE TOWN

DEPARTMENT OF ELECTRICAL ENGINEERING

EEE221W(A) & EEE225F(A) - Electrical and Electronic Circuits

EXAMINATION

21 June 1996

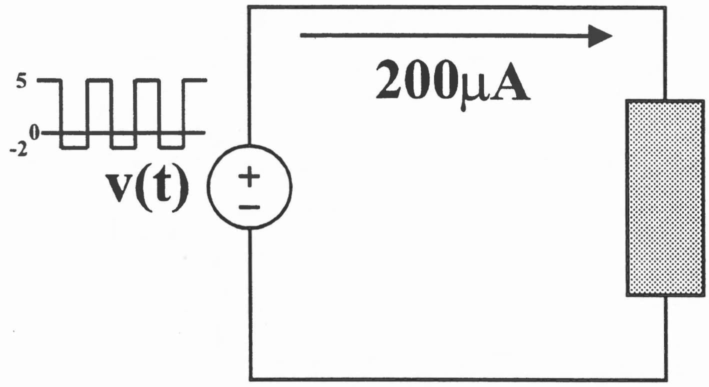
- Attempt a total of **TEN** questions, choosing **either six or seven** of the C-questions (on electrical circuit analysis) and then **either three or four** of the E-questions (on basic electronics).
- Write all necessary working on the paper, and give your answers to 3 significant figures in the spaces provided, always adding **engineering units** where appropriate.
- You should always **explain your methods** with care, and use **neat, labelled diagrams** where appropriate. Some credit can be given for promising working, even if the final answer is incorrect.
- All questions carry equal marks (10 marks each). **100 marks gains 100%**.
- The time allowed is **3 hours**.

Question C1 (10 MARKS)

The device shown in the circuit diagram below is supplied by a voltage source whose output voltage is a square-wave described by:

$$v(t) = \begin{cases} 5V & 0 < t < \frac{T}{2} \\ -2V & \frac{T}{2} < t < T \end{cases} \quad \text{where} \quad v(t+T) = v(t) \quad \text{and} \quad T = 1\text{ms}.$$

Despite the variations in the supply voltage, the device draws a steady current of 200μA.



Find:

(a) The total charge that enters the device from the source between $t = 0$ and $t = 1.4$ms.

$Q_{tot} =$

(b) The rate at which energy is being delivered by the source to the device at $t = 1.4$ms.

$dw/dt =$

(c) The average power delivered by the source to the device.

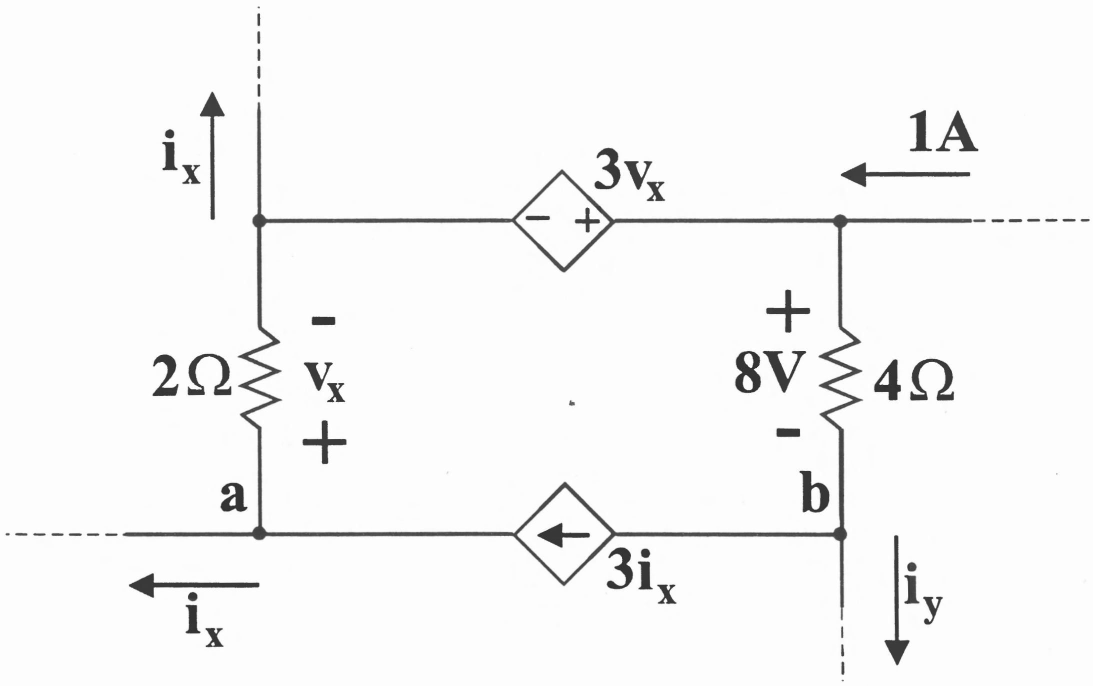
$P =$

(d) The rms value of the source voltage.

$V_{rms} =$

Question C2 (10 MARKS)

Find i_y and v_{ab} in the circuit fragment shown below.

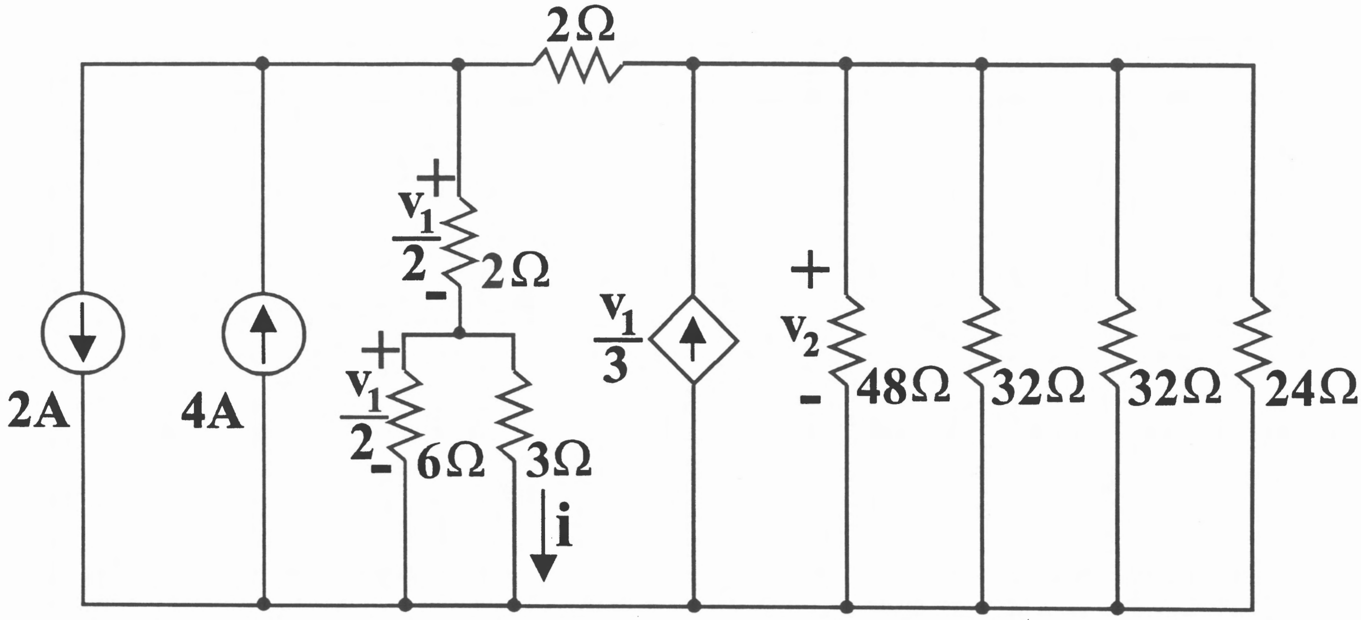


$i_y =$

$v_{ab} =$

Question C3 (10 MARKS)

Simplify the circuit below until it consists of three resistors and two current sources. Then use nodal analysis to solve your simplified circuit and hence to find the current i.



Simplified circuit

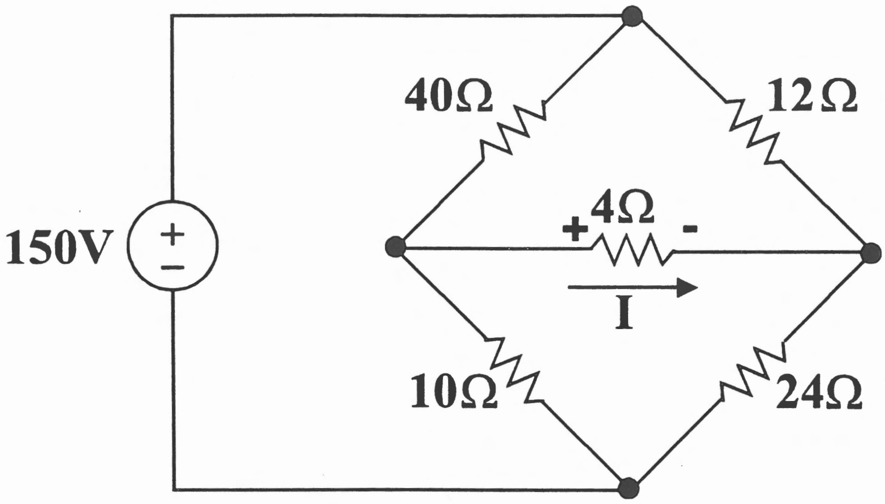
Nodal analysis

. . . .

$i =$

Question C4 (10 MARKS)

Redraw the circuit below with everything except the 4Ω resistor replaced by its Thévenin equivalent, and hence find the value of the current I.



....

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....

$R_{th} =$		$v_{oc} =$

Thévenin equivalent circuit

....

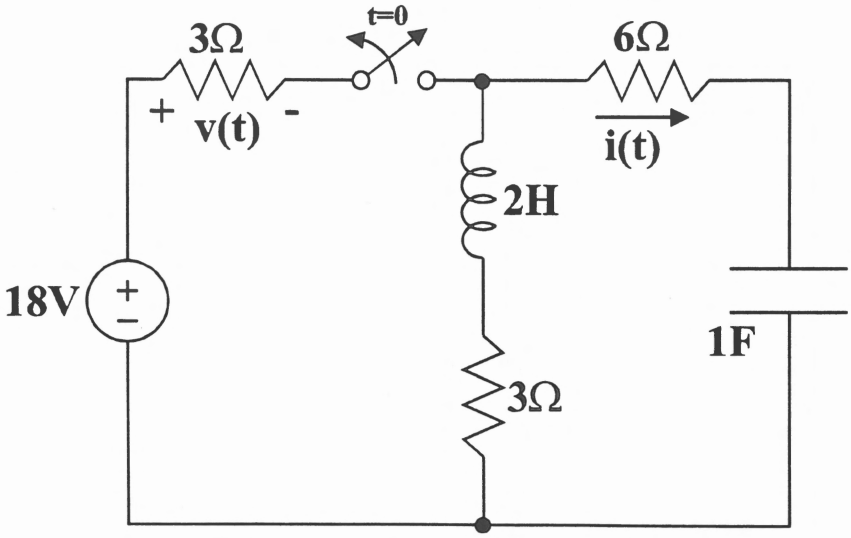
....

....

....

.... $I =$

Question C5 (10 MARKS)



At a certain instant, $v(0^-) = 9$V and $i(0^-) = 1$A in the circuit above. The switch is then opened at time $t = 0$. Find, an instant later, the values of $i(0^+)$ and $di(0^+)/dt$.

$i(0^+) =$

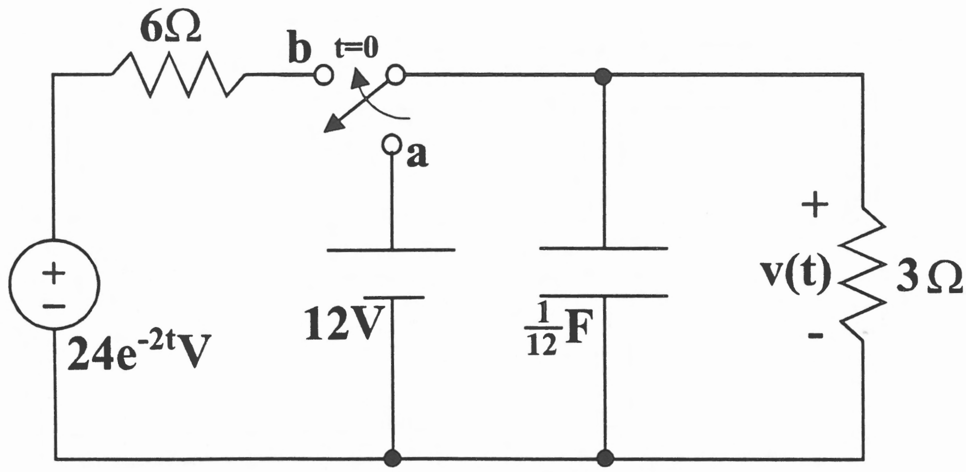
$di(0^+)/dt =$

After the switch has been open for a long time, the clock re-starts at $t = 0$ and the switch is closed again. Find, an instant later, the value of $v(0^+)$.

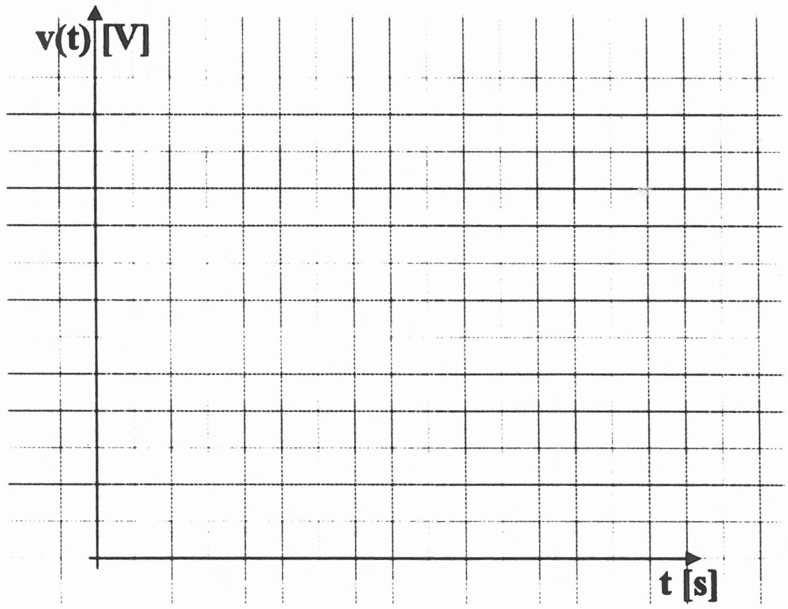
$v(0^+) =$

Question C6 (10 MARKS)

At time $t = 0$, the switch in the circuit below moves from position a to position b. Calculate and graph $v(t)$ for $t > 0$, indicating all significant values such as τ and $v(\tau)$ on the axes of your graph.

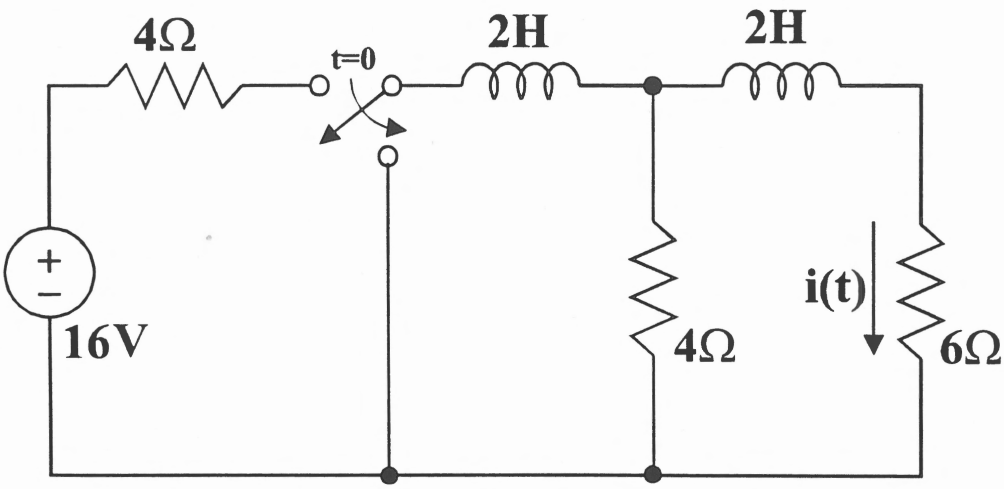


$v(t) =$



Question C7 (10 MARKS)

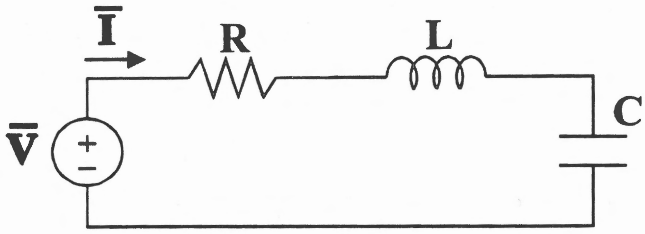
For the circuit below, find the full expression for $i(t)$ after $t = 0$, which is the instant at which the switch changes position. You are given that the circuit is in the dc steady state at $t = 0^-$.



$i(t) =$

Question C8 (10 MARKS)

Use frequency-domain analysis to find $\mathbf{H}(j\omega)$ and hence $|\mathbf{H}(j\omega)|$ for the series RLC circuit shown below. You should regard $\mathbf{I}$ as the circuit's output and $\mathbf{V}$ as its input.

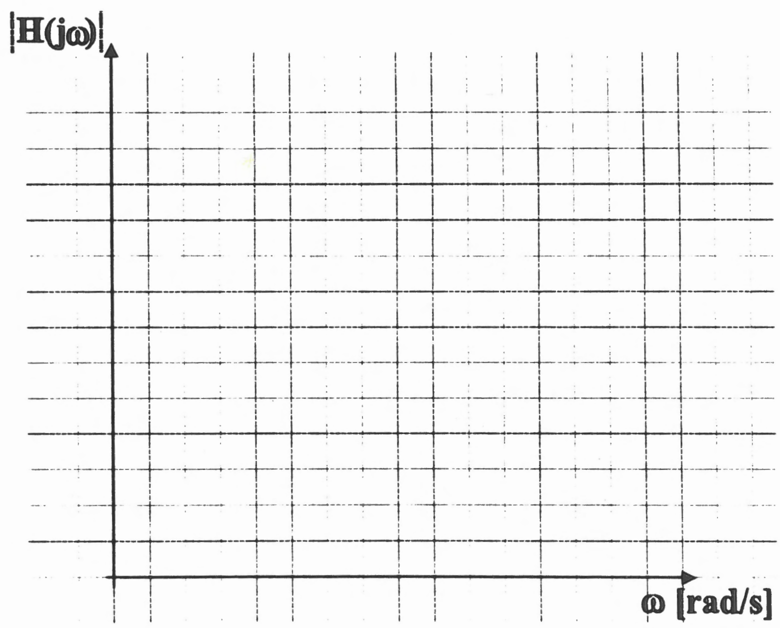


$\mathbf{H}(j\omega) =$	$\|\mathbf{H}(j\omega)\| =$

Now sketch the graph of the transfer function that you have found, and show on your sketch (or otherwise explain) what is meant by:

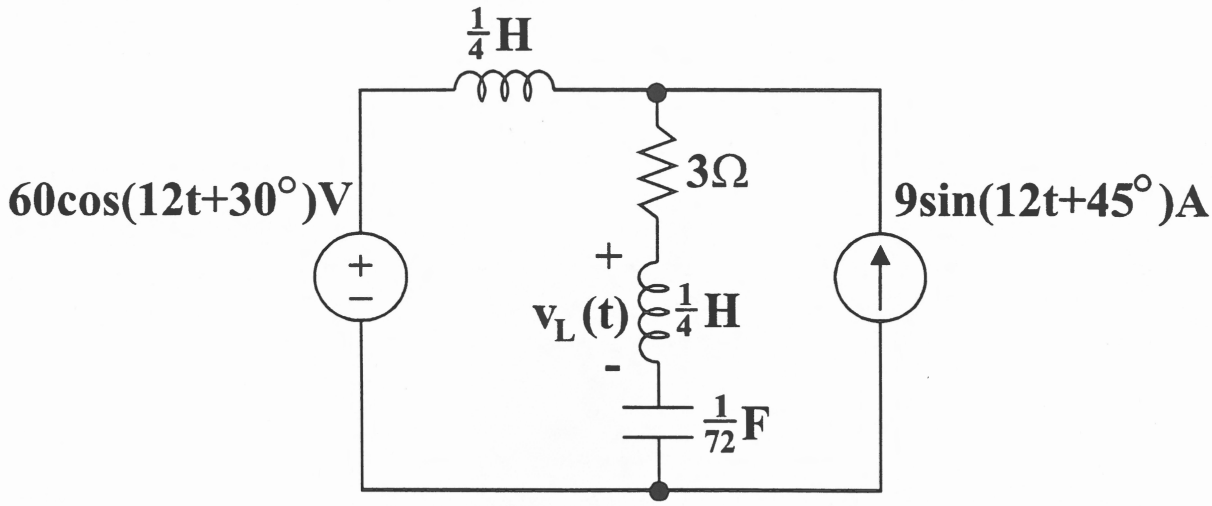
(a) the bandwidth (b) the resonant frequency (c) the quality factor

Explanation



Question C9 (10 MARKS)

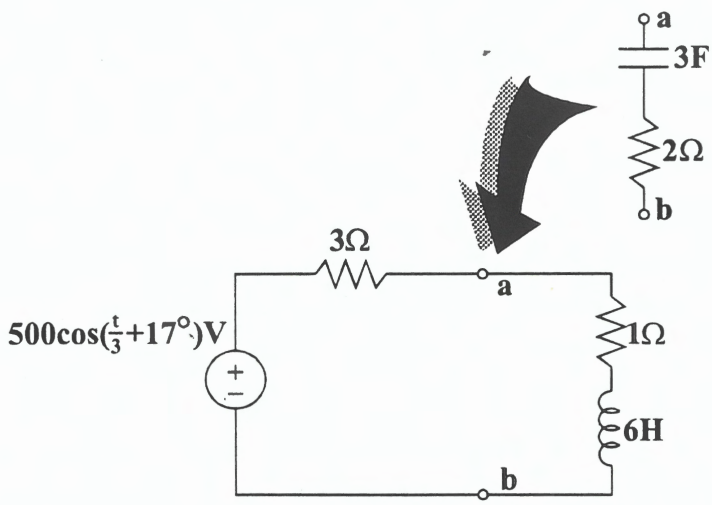
Use the Principle of Superposition in the phasor equivalent of the circuit below to calculate the ac steady-state voltage, $v_L(t)$, which is across the inductor as shown. To make your phasor equivalent circuit, simply add the necessary information about impedances and sources to the given diagram.



$v_L(t) =$

Question C10 (10 MARKS)

In the circuit below, find the power factor seen from the source terminals and the average power delivered to the circuit by the source. Write each element's complex impedance on the given diagram.



. . . .

$pf =$

$P =$

A 3F capacitor that has a 2Ω resistor in series with it, as shown above, is now connected between points a and b. Find the new power factor seen from the source terminals, and the average power now delivered to the circuit by the source.

. . . .

$pf =$

$P =$

State one advantage and one disadvantage of the new connection.

Advantage . . . .

Disadvantage . . . .